MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXXIX. DECEMBER 14, 1928. No. 2

Rev. T. E. R. PHILLIPS, PRESIDENT, in the Chair.

The President referred to the death of Dr. J. W. L. Glaisher, and read the following resolution of the Council :—

"The President, Council, and Fellows of the Royal Astronomical Society wish to put on record their deep sense of the loss the Society has sustained in the death of Dr. J. W. L. Glaisher, for fifty-five years member of the Council, during which time he was Secretary, President (on two occasions), and frequently Vice-President."

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

The Rev. William Ira Brown, Rishton Vicarage, Blackburn, Lancs. (proposed by F. J. Wright) ;

Arthur Kington Chapman, Ellerslie, Auchenflower, Queensland, Australia (proposed by D. Eglinton) ;

Egbert Adriaan Kreiken, Ph.D., Bosscha-Sterrewacht, Lembang, Java, Dutch East Indies (proposed by W. M. H. Greaves) ;

Stavros Plakidis, National Observatory, Athens (proposed by L. J. Comrie) ;

John Herbert Hedley Robinson, 17 Elmgrove Road, Cotham, Bristol (proposed by T. L. Evans) ;

Harlan T. Stetson, Professor of Astronomy, Harvard University, Cambridge, Mass., U.S.A. (proposed by F. J. M. Stratton);
Reginald Owen Street, M.A., M.Sc., The University, Liverpool (proposed by J. Proudman);
R. v. d. R. Woolley, B.A., Gonville and Caius College, Cambridge (proposed by F. J. M. Stratton); and
Herman Zanstra, Ph.D., Imperial College of Science and Technology, London, S.W. 7 (proposed by S. Chapman).

Seventy-four presents were announced as having been received since the last meeting, including amongst others :—

Cambridge Observatory, Observations, vol. xxiv., pt. 2, vol. xxvi.
Greenwich, Royal Observatory, Observations made with the Cookson Floating Telescope, 1919–27. Variation of Latitude and the Constant of Aberration.
Göttingen Observatory, Astronomische Mitteilungen, Theil 17–20.
———————— Veröffentlichungen, Heft 1–3.
Hamburg Observatory, Erstes Bergedorfer Sternverzeichniss, 1925.
———————— Index der Sternörter, 1900–25, 2 vols.
International Astronomical Union, Bulletin for Character Figures of Solar Phenomena, No. 1.
Upsala Observatory, Map of the Milky Way Structure drawn according to Bailey's photographs by O. Jahnke under the supervision of K. Lundmark.
J. Voûte, Second Catalogue of Radial Velocities.

The President announced that the late Mr. W. J. Thorrowgood had bequeathed to the Society his 8-inch Refractor by Cooke, together with its Observatory and a sum of money to cover the expenses of removal.

Mr. C. O. Bartrum and Mr. T. Whitwell were appointed auditors of the Treasurer's accounts for 1928.

The Analysis of Nechvile's Proper Motions.

By W. M. Smart, D.Sc., and H. E. Green, M.A.

(*Communicated by the Director, Cambridge Observatory.*)

§ 1. *Summary.*—The proper motions recently measured by Dr. V. Nechvile of Prague have been analysed in order to determine the systematic motions of the stars. Nechvile's published proper motions have a minimum value of 2″ per century, and consequently the interpretation of the frequency curves requires a knowledge of the distribution of the stars in space. Dyson's law of stellar density has been assumed, and the true frequency distributions of the proper motions in the several regions of the sky considered have been deduced. The calculation of the drift and stream constants follows. The principal results are as follows :—

(a) *The Drift Constants.*

	Drift I.	Drift II.
R.A. of apex	92°	265°
Dec. ,,	0°	− 53°
Drift-velocity	1·34	0·91

(b) *The Vertex of Star-streaming.*

R.A. 90° ; Dec. + 21°.

(c) *The Constants of the Solar Motion.*

R.A. of apex of solar motion	277°
Dec. ,, ,, ,,	+ 42°
Velocity	0·54

§ 2. In 1885 the brothers Henry of Paris Observatory had obtained their first successful plates with their newly designed refractor—the prototype of the astrographic telescope,—and in the same year the photographic survey of the southern sky was inaugurated by Sir David Gill at the Cape. This year was thus a memorable one in the history of astronomy. These early Paris plates are of very great value as they record the positions of stars as faint as the 16th magnitude at the most remote photographic epoch. Twenty-two of the regions photographed by MM. Henry between 1886 and 1889 have been rephotographed by Dr. V. Nechvile * at Paris with the same instrument between 1922 and 1924. There is thus, on the average, an interval of about 35 years between the two sets of plates, ample enough for the accurate determination of the proper motions. In Nechvile's investigation, all proper motions (with a few unimportant exceptions) less than 2″ per

* "Recherches sur les Mouvements Propres de 3802 étoiles," *Publications de l'Observatoire national de Prague*, No. 4.

century are omitted from the published material; his proper motions are thus of the restricted kind with which one of us dealt in general terms in a previous paper.* It is this particular feature of Nechvile's proper motions that has suggested the analysis of the systematic motions of the stars, in combination with an assumed law of stellar density.

§ 3. Nechvile's memoir contains the frequency curves for each region—these have been obtained by counting the number of stars moving in 10° sectors of position-angle and smoothing the counts by taking the means of three adjacent sectors. We have grouped together adjacent regions wherever such a procedure was possible; Table I. shows the details.

TABLE I.

Group.	Regions.	R.A. h m	Dec. °	No. of Stars.	Gal. Lat. °	Weight.
A	1–8	0 38	65·6	938	+ 5	3
B	10	8 34	20·3	182	+33	1
C	11–13	14 4	56·0	464	+60	2
D	14, 15	16 3	56·1	427	+45	2
E	16–18	18 47	32·8	736	+13	3
F	19–22	20 31	39·5	954	0	3

The second column contains the region-numbers in Nechvile's paper which have been combined to form the single groups A to F. (Region No. 9 has been omitted owing to the small number of stars with measured proper motions.) The third and fourth columns give the centres of the groups weighted according to the numbers of stars in the component regions. The last column gives the weights assigned to the various groups in the subsequent analysis.

§ 4. Nechvile does not give in his paper the counts in each region of the proper motions in the different position-angle sectors, but only the frequency curves derived from these counts. It is, however, a simple matter to deduce from his diagrams the numbers of proper motions in the various sectors. In Table II. these numbers are given for the six groups in the columns headed by ρ (fractional values occur since the numbers have been smoothed by taking as the effective number, moving in the sector $\theta - 5°$ to $\theta + 5°$, the mean of the observed numbers in this sector and its two flanking sectors). The frequency curves based on the values of ρ in Table II. have been analysed into two drift curves. If R_1 and R_2 are the numbers of stars in these drifts, θ_1 and θ_2 the position-angles in which the drifts are pointing, and hV_1', hV_2' the drift-velocities, then the theoretical number r of stars moving in the sector $\theta - 5°$ to $\theta + 5°$ is given † by

$$r = 0{\cdot}0492R_1\phi(hV_1', \theta - \theta_1) + 0{\cdot}0492R_2\phi(hV_2', \theta - \theta_2) . \quad (1)$$

* W. M. Smart, *M.N.*, **89**, 93: "On the Frequency Distribution of restricted Proper Motions."

† *M.N.*, **87**, 127.

TABLE II.

Position Angle.	A. (938 Stars.)		B. (182 Stars.)		C. (464 Stars.)		D. (427 Stars.)		E. (736 Stars.)		F. (954 Stars.)	
	ρ.	Δρ.	ρ.	Δρ.	ρ.	Δρ.	ρ.	Δρ.	ρ.	Δρ.	ρ.	Δρ.
°												
5	8·0	+5	3·0	+2	9·3	+4	11·7	0	32·3	+2	27·7	0
15	10·3	+7	2·3	+2	5·0	0	7·7	−1	34·7	+1	37·0	−1
25	12·3	+7	2·3	+1	3·0	−1	3·0	−3	32·0	−1	50·0	0
35	14·3	+6	1·0	0	2·7	−2	3·3	−1	29·3	−1	61·0	+1
45	19·3	+4	1·0	0	3·7	−1	3·3	0	23·3	−3	60·7	−2
55	23·7	−2	1·7	+1	4·7	0	3·3	0	15·7	−4	59·0	−1
65	39·7	−4	1·3	0	6·0	0	2·3	0	12·5	−3	49·0	−1
75	57·3	−9	2·7	+1	7·7	+1	2·0	−1	10·3	−1	36·7	−2
85	89·3	+4	2·7	0	8·7	0	1·7	−2	9·7	+1	26·7	−1
95	96·3	+2	4·0	+1	9·3	−1	9·0	+4	7·3	0	21·7	+2
105	92·7	+7	3·3	−1	9·0	−3	3·3	−2	7·0	0	17·3	+3
115	68·0	+1	5·7	0	9·7	−4	4·3	−3	6·7	−1	12·0	+1
125	46·3	+2	5·3	−2	17·3	+2	9·3	−1	7·3	−2	11·3	+1
135	33·3	+6	8·3	0	19·7	+3	12·3	0	11·7	−1	9·7	−2
145	23·0	+8	10·0	0	20·7	+4	16·0	0	18·7	+1	13·7	−1
155	16·3	+7	12·3	+2	14·7	−2	19·7	+1	25·0	+1	16·7	−4
165	11·3	+9	11·7	+1	15·0	−1	22·3	+2	31·0	−1	22·0	−6
175	9·3	+4	10·7	0	15·3	+1	23·3	+2	41·3	+2	40·0	+3
185	12·0	+5	10·0	0	13·0	0	19·3	−1	44·3	0	48·0	+2
195	13·3	+5	7·7	−2	10·3	−1	16·0	−2	49·3	+3	51·3	+1
205	12·7	+2	7·0	−2	7·7	−3	14·0	−2	45·3	+1	49·0	−1
215	12·0	−3	7·3	−2	9·3	−1	13·3	0	41·3	+2	47·0	+2
225	13·7	−5	7·7	−2	11·0	+1	12·3	+2	31·0	−1	44·3	+7
235	18·0	−6	9·3	−1	13·0	+1	12·3	+3	21·7	−2	29·3	+1
245	23·7	−4	10·3	0	16·0	+2	10·7	+3	16·7	−1	19·3	0
255	30·3	0	9·0	0	22·3	+5	12·0	+4	13·3	+1	10·0	−4
265	32·3	+2	7·3	0	22·0	+1	11·7	+4	11·3	+2	9·0	0
275	28·7	+1	3·7	−3	23·3	−1	11·0	+1	9·0	+2	8·0	+1
285	22·0	−2	3·7	−1	20·0	−7	10·7	−2	8·0	+2	6·7	+2
295	13·7	−5	1·7	−1	22·0	−4	11·7	−4	7·9	+1	6·3	+2
305	9·0	−6	1·7	0	24·3	+1	17·0	−2	8·3	+2	5·7	+2
315	4·7	−6	1·3	0	22·7	+3	20·7	0	9·7	+1	5·7	+2
325	2·7	−5	1·7	+1	17·7	+2	22·7	+1	11·0	0	5·7	0
335	2·7	−3	1·3	+1	10·7	−2	22·0	+1	13·0	−2	7·3	0
345	4·0	0	1·3	+1	10·0	+1	18·7	0	14·7	−5	9·7	−2
355	4·3	+1	2·0	+1	9·7	+2	18·7	+4	26·3	+1	20·7	+3

In Table II. the entries under $\Delta\rho$ are the differences (to the nearest unit) between the observed values ρ in the different sectors and the numbers r derived from (1) by choosing the drift constants in such a way as to give the best fit between the observed and theoretical frequency curves. The entries under $\Delta\rho$ indicate that, in general, the observed frequency curves can be represented very satisfactorily by the theoretical curves obtained from (1).

§ 5. It has to be remembered that Nechvile's proper motions are exclusive of values less than 2″ per century, and that consequently the drift constants derived from the preceding analysis do not refer to the true drifts. It has been shown * that, on the basis of Dyson's law of stellar density, the frequency distribution of proper motions, which are restricted in a manner similar to that in which Nechvile's proper motions are restricted, differs inappreciably from a true drift distribution—defined with reference to *all* values of the proper motions—with a substantially greater velocity than that of the real drift. Thus, to take an example, the analysis of one of Nechvile's groups shows that the distribution of proper motions can be represented by means of two pseudo-drifts with velocities 1·7 and 1·0 respectively, the corresponding number of stars being 590 and 308 (these are the values of $h\mathrm{V}_1'$, $h\mathrm{V}_2'$, and R_1, R_2 for group A; see Table IV. below); in particular, the values of $h\mathrm{V}_1'$ and $h\mathrm{V}_2'$ are not the true drift velocities which would presumably have been found if all proper motions less than 2″ per century had been included in the observational material. The problem is to derive the true drifts from the pseudo-drifts obtained from the analysis of the proper motions exceeding 2″ per century. Dyson's density function $\mathrm{D}(r)$ in the form used in the theory is given by

$$\mathrm{D}(r) = \frac{\mathrm{A}}{r} e^{-h^2k^2r^2} \quad . \quad . \quad . \quad . \quad (2)$$

in which k may be presumed to be dependent on the galactic latitude of the region concerned. The Cambridge proper motions gave some indication of the way k varies with galactic latitude, and it is thus possible to assign to each of the groups A–F an appropriate value of k which it is believed will be a good approximation. Actually, for small values of k —corresponding to regions of low galactic latitude—precision is of little consequence for the needs of the present discussion, as the following table † will show. Here $h\mathrm{V}$ is the true drift velocity and the entries in the table are the pseudo-drift velocities $h\mathrm{V}'$ (for different values of k) when the centennial proper motions are restricted to values of 2″ and over.

The analysis of Nechvile's proper motions furnishes us with the values of $h\mathrm{V}'$—for each of the two pseudo-drifts—and k being supposed known, Table III. enables us to deduce the appropriate values of $h\mathrm{V}$—for each of the true drifts—with more than the necessary accuracy.

* W. M. Smart, *M.N.*, **89**, 93.

† This table is reproduced from *M.N.*, **89**, 98.

TABLE III.

Values of kV′.

		kV.			
k.	kV→	0·0.	0·5.	1·0.	1·5.
2·0		0·0	0·60	1·10	1·58
1·0		0·0	0·66	1·22	1·65
0·5		0·0	0·68	1·25	1·71
0·0		0·0	0·70	1·26	1·73

In Table IV. are given the results of analysing Nechvile's frequency curves—these results are the values of $k\mathrm{V}_1'$ and $k\mathrm{V}_2'$ (the pseudo-drift velocities), the position angles θ_1 and θ_2 in which the drifts are pointing, and R_1 and R_2 the numbers of stars, with centennial proper motions of 2″ and over. Also the table gives the adopted values of k (depending on galactic latitude) and the deduced values of the true drift velocities $k\mathrm{V}_1$ and $k\mathrm{V}_2$ obtained from the data of Table III.

TABLE IV.

The Drift Constants.

Group.	Galactic Latitude.	k.	Drift I.				Drift II.			
			$k\mathrm{V}_1'$.	$k\mathrm{V}_1$.	θ_1.	R_1.	$k\mathrm{V}_2'$.	$k\mathrm{V}_2$.	θ_2.	R_2.
A	+ 5°	0·4	1·7	1·5	95°	590	1·0	0·75	260°	308
B	+33	1·0	1·3	1·15	245	68	0·9	0·7	160	114
C	+60	1·6	1·15	1·0	290	230	0·65	0·55	145	234
D	+45	1·3	1·1	0·95	325	197	0·9	0·75	175	230
E	+13	0·6	1·1	0·9	18	307	1·1	0·9	195	429
F	0	0·2	1·3	1·1	45	509	1·2	1·0	200	445

§ 6. Let X, Y, Z denote the components, referred to equatorial rectangular axes, of the linear velocity of a drift, and (α, δ) the co-ordinates of the centre of a group (the latter are given in Table I.). Then each group provides the pair of equations in X, Y, and Z with reference to Drift I. :

$$-\mathrm{X}\sin\alpha + \mathrm{Y}\cos\alpha = k\mathrm{V}_1\sin\theta_1 \quad . \quad (3)$$

$$-\mathrm{X}\cos\alpha\sin\delta - \mathrm{Y}\sin\alpha\sin\delta + \mathrm{Z}\cos\delta = k\mathrm{V}_1\cos\theta_1 \quad . \quad (4)$$

the values of $k\mathrm{V}_1$ and θ_1 being found in Table IV. There are similar equations for Drift II. The equations (3) and (4) have been solved by least squares ; the results are as follows :—

	Drift I.		Drift II.	
X	−0·039	±0·049	−0·049	±0·044
Y	+1·343	±0·059	−0·545	±0·054
Z	+0·072	±0·072	−0·725	±0·065

These velocities are, of course, relative to the sun. From the values of X, Y, and Z just stated, the co-ordinates of the drift-apices are easily obtained. Also, if hW denotes the linear space velocity (relative to the sun) of a drift, we have

$$h\mathrm{W} = (\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad (5)$$

The drift-apices and velocities are as follows :—

	Drift I.		Drift II.	
R.A. of apex	91°·7	±2°·1	264°·9	± 4°·6
Dec. ,,	+0°·4	±3°·1	−53°·0	± 3°·7
Space-velocity (hW)	1·344	±0·059	0·908	±0·061

§ 7. Nechvile has made a determination of the positions of the drift-apices by Airy's method ; the relevant equations are of the form of (3) and (4) with the mean value of μ_α and of $\mu\delta$ on the right-hand sides. It is first necessary in his procedure to assign any one star either to Drift I. or to Drift II., and although the drifts are well separated in his regions the method cannot be free from uncertainties. A more serious objection, however, is the use of the means of the proper motions of stars, ranging from magnitude 6 to magnitude 16, in Airy's equations. Nechvile's results are in surprisingly good agreement with the results of our analysis ; they are found in Table V. below, which contains the results of several investigations for comparison with the present work.

TABLE V.

	Drift I.			Drift II.		
	Apex.		Velocity.	Apex.		Velocity
	α.	δ.	(hW).	α.	δ.	(hW).
	°			°	°	
Eddington (Boss meridian P.M.'s)	91	−15	1·52	288	−64	0·86
Smart (Cambridge photographic P.M.'s)	88	−12	1·50	289	−73	0·97
Smart (Groningen P.M.'s)	89	− 3	1·25	279	−56	0·61
Nechvile (Airy's method)	94	− 2	..	266	−47	..
Smart and Green (present paper)	92	0	1·34	265	−53	0·91

§ 8. Using our position of Drift I. apex stated in § 6 we can calculate the position-angle θ_1 of the drift for each of the six groups, and then make a comparison between the values so calculated and the values derived from the actual analysis of the individual groups. Let $\Delta\theta_1$ denote the differences in θ_1 in the sense " observed minus calculated " ; the values of $\Delta\theta_1$ are given in the second column of Table VI. below. Again, the true value of the drift velocity $h\mathrm{V}_1$ in any region is the projection on the tangent plane of the region of the space velocity hW of the drift ; hence if λ_1 is the angular distance of the centre of the region from Drift I. apex, we obtain

$$h\mathrm{V}_1 = h\mathrm{W} \sin \lambda_1 \quad . \quad . \quad . \quad . \quad (6)$$

The value of hW is known from the results in § 6, λ_1 can be calculated for each group (the values are in the third column of Table VI.), and thus we can obtain the calculated values of $h\mathrm{V}_1$. Let $\Delta(h\mathrm{V}_1)$ denote the difference between the observed values of $h\mathrm{V}_1$ (from Table IV.). and the calculated values. The fourth column of Table VI. contains the values of $\Delta(h\mathrm{V}_1)$. We have also calculated the shortest angular

TABLE VI.

	Drift I.				Drift II.			
Group.	$\Delta\theta_1$.	λ_1.	$\Delta(h\mathrm{V}_1)$.	D_1.	$\Delta\theta_2$.	λ_2.	$\Delta(h\mathrm{V}_2)$.	D_2
	°	°		°	°	°		°
A	−2	86	+0·16	2	+14	139	+0·16	9
B	−1	41	+0·27	0	+15	133	+0·09	11
C	0	105	−0·29	0	− 6	118	−0·25	5
D	+1	118	−0·23	1	+10	111	−0·10	9
E	−6	145	+0·14	3	+ 5	87	0·00	5
F	−3	128	+0·05	3	− 5	100	+0·10	4

distances D_1 from Drift I. apex to the great circles defined by the observed values of θ_1; this is perhaps the best criterion for exhibiting the convergency towards the apex of the great circles defined by θ_1; the values of D_1 are in the fifth column. The remainder of Table VI. gives the results of similar calculations for Drift II.

It will be noticed from the values of $\Delta\theta_1$ and D_1 in Table VI. that the convergency of the great circles defined by the observed values of θ_1 is very satisfactory. The convergency is not so perfect for Drift II., but this is inevitable, as owing to the flatter curves of this drift the position-angle θ_2 corresponding to the maximum frequency of Drift II. stars cannot be determined with the same delicacy generally as for the more pointed curves of Drift I. The residuals $\Delta(h\mathrm{V}_2)$ for Drift II. are satisfactory, but the column $\Delta(h\mathrm{V}_1)$ shows some unexpectedly high differences.

§ 9. We now calculate the vertex of star-streaming. Let (ξ, η, ζ) denote the velocity of Drift I. relative to Drift II. Then from the values of X, Y, Z in § 6 we obtain

$$\begin{aligned} \xi &\equiv \mathrm{X}_1 - \mathrm{X}_2 = +0{\cdot}010, \\ \eta &\equiv \mathrm{Y}_1 - \mathrm{Y}_2 = +1{\cdot}888, \\ \zeta &\equiv \mathrm{Z}_1 - \mathrm{Z}_2 = +0{\cdot}735, \end{aligned}$$

and from these we derive the right ascension and declination of the vertex and the velocity hU (in terms of the theoretical unit $1/h$) of one stream relatively to the other. The results are in Table VII., together with the results of some other investigations.

Nechvile's investigation does not throw any light on the value of hU. The position of the vertex which we have just found (last line of Table VII.) is only 1° from the galactic equator—the galactic longitude is 156°. The value of hU which we have derived from this investigation is a little higher than that generally found.

TABLE VII.

The Vertex of Star-Streaming.

	Vertex.		Velocity.
	α.	δ.	kU.
	°	°	
Eddington (Boss meridian P.M.'s)	94	+12	1·87
Smart (Cambridge photographic P.M.'s)	92	+20	1·84
Smart (Groningen P.M.'s)	91	+15	1·64
Smart and Green (Nechvile's P.M.'s)	90	+21	2·03

We have also made a solution with the uncorrected values $k\mathrm{V}_1'$ and $k\mathrm{V}_2'$ of the pseudo-drift velocities which, we recall, are derived from the direct analysis of the frequency curves of the restricted proper motions. The co-ordinates of the vertex are practically identical with those in the last line of Table VII., but the velocity of one stream relatively to the other has the very high value 2·40.

It would appear that on the whole the corrections, based on the theoretical frequency curves of the restricted proper motions and applied in Table IV., are approximately of the right order of magnitude.

§ 10. The question of the solar motion can be approached, so far as this investigation is concerned, in two ways. We can first of all define the solar motion with reference to the stars with proper motions not less than 2″ per century, and, secondly, we can define it with reference to all the stars without restricting the values of the proper motions in any way. The second is the one at which to aim, but it looks, at first sight, that it is outside the bounds of practical calculation for the simple reason that we have no knowledge—we are referring to Nechvile's memoir—of the numbers of stars with centennial proper motions between 0″ and 2″. We should naturally expect, however, that the two methods of deriving the apex of the solar motion would lead to very much the same result; the velocity of the sun, however, must depend on the selection of stars with reference to which it is defined.

Let us consider the solar motion defined with reference to stars with centennial proper motions not less than 2″. Our analysis of the six groups gives us the values of $k\mathrm{V}_1'$, $k\mathrm{V}_2'$, θ_1, θ_2, and R_1, R_2 (Table IV.) —these are the constants of what we have called the pseudo-drifts. Then by solving equations (3) and (4), in which we suppose the values of $k\mathrm{V}_1'$, $k\mathrm{V}_2'$ substituted on the right-hand sides, we obtain the positions of the apices and the velocities of the pseudo-drifts. Table IV. shows that, of the 3661 stars in the six groups with centennial proper motions of 2″ and over, 1901 are allocated to Drift I. and 1960 to Drift II. The position of the solar apex and the solar velocity can now be obtained—the results are:

Solar apex	294°·8, + 39°·7.
Solar velocity	0·604.

This value of the solar velocity cannot be compared directly with the values derived in investigations which deal with true drifts owing to

the selection of stars with centennial proper motions restricted to 2″ and over.

We shall now make use of our data to derive the solar apex and the solar velocity defined with reference to the true drifts. Consider Drift I. for a particular region. We have found by analysing the frequency curves the number R_1 of stars belonging to the pseudo-drift, with proper motions of 2″ and over. If N_1 is the total number of stars in the true drift with proper motions from 0 to ∞, we have *

$$R_1 = N_1[1 - \eta^2 e^{-h^2V_1^2(1-\eta^2)}] \quad . \quad . \quad . \quad (7)$$

where hV_1 is the velocity of the true Drift I., and η is defined in terms of the density constant k and the limiting proper motion μ (in this instance, 2″ per century) by

$$\eta = \frac{\mu}{(\mu^2 + k^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (8)$$

Thus, since R_1 is obtained from the observations, N_1 can be computed by means of (7) and (8). Similarly, N_2 can be found. Thus the complete specification of the six drift constants for any region can be obtained; the analysis of the frequency curves gives us θ_1, θ_2, and hV_1', hV_2', and from the latter by applying the corrections as in § 5 we deduce hV_1 and hV_2; and we have just shown how the remaining constants N_1, N_2 of the true drifts can be derived. When k is small, as it is in regions close to the galactic plane, the value of η in (8) is close to unity and so the coefficient of N_1 in (7) becomes very small; in this case it is easily shown that, with sufficient accuracy,

$$R_1 = N_1 \cdot \frac{k^2}{4}(1 + h^2V_1^2) \quad . \quad . \quad . \quad (9)$$

There is a similar equation connecting R_2 and N_2.

Hence it appears that unless the value of k is known accurately, the value of N_1 cannot be deduced without grave errors; and this, of course, is equally true of N_2. In our preceding work (see Table IV.) the values of k have been estimated for the various groups, and for the purposes then in view these values, admittedly approximate, are nevertheless sufficiently accurate to provide the desired information. But approximate values of k are of little use in the present connection (that is when k is small) and we have to fall back on the calculation of the ratio of $N_1 : N_2$. Assuming that k is the same for the two drifts—in other words, that the mean distance of Drift I. stars is the same as the mean distance of Drift II. stars—we have from (9) and its companion equation

$$\frac{N_1}{N_2} = \frac{R_1}{R_2} \cdot \frac{1 + h^2V_2^2}{1 + h^2V_1^2} \quad . \quad . \quad . \quad (10)$$

It is convenient to express N_1 and N_2 as percentages of $(N_1 + N_2)$, and these are shown in Table VIII. for the six regions. When the assumed value of k is small, the calculations have been made according

* *M.N.*, **88**, 580.

to (10) and in the other groups according to (7), and the corresponding equation for the second drift.

TABLE VIII.

Group.	N_1.	N_2.	Weight.
A	49	51	3
B	30	70	1
C	44	56	2
D	43	57	2
E	42	58	3
F	41	59	3

The weights assigned are the same as in Table I. From the data of Table VIII. it is found that 45 per cent. of the stars belong to Drift I. and 55 per cent. to Drift II. This can only be regarded as an approximate result, depending as it does on an imperfect system of weighting and on the assumption that the stars of Drift I. are at the same mean distance as the stars of Drift II. For our further calculations we have taken the numerical strengths of the two streams to be equal—in line with other investigations. It is to be remembered that the streams with which we are now dealing consist of stars with *all* values of proper motions. Our results for the solar apex and the solar motion (the latter in terms of the theoretical unit $1/h$) are found in the last line of Table IX., which includes several other results for purposes of comparison.

TABLE IX.

	Solar Apex.		Solar Velocity.
	α.	δ.	
	°	°	
Boss (meridian P.M.'s)	270	+34	..
Eddington (meridian P.M.'s)	267	+36	0·91
Smart (Cambridge P.M.'s)	273	+44	0·88
Smart (Groningen P.M.'s)	273	+31	0·55
Smart and Green (present paper)	277	+42	0·54

The position of the solar apex which we have just found is in very good agreement with the position found from the Cambridge photographic proper motions,* but the solar velocity is considerably less than that found for the Cambridge stars; the value of the solar motion obtained from Nechvile's proper motions, however, is almost exactly that derived from the analysis of the Groningen proper motions.

§ 11. In the preceding pages we have applied an assumed law of stellar density to the problem presented by the frequency distributions of restricted proper motions, and have derived results relating to the positions of the drift-apices, the drift velocities, the vertex of star-streaming, and the constants of the solar motion which are, on the

* *Cambridge Observations*, xxvi; and *M.N.*, **87**, 137.

whole, in satisfactory agreement with results of investigations in which there are no artificial restrictions of the proper motions as regards the magnitudes of the latter. Our conclusions from this work cannot, of course, be interpreted as a proof that the basic density law of the analysis contains essentially the whole truth concerning stellar distribution; our results, however, may be considered as lending some support to the thesis, which is sustained fairly adequately by other investigations, that the assumed density law is in any event a good first approximation.

Observatory, Cambridge:
1928 *December* 1.

Ionization in Stellar Atmospheres. Part II. Absolute Magnitude Effects. By E. A. Milne, F.R.S.

1. *Scope of the Paper.*—In Part I. of this paper * formulæ have been obtained for the total number N_r of atoms per cm.^{2} in a stellar atmosphere, in a given state of ionization r, down to the level where the electron-pressure is P, allowing accurately for changes of ionization from level to level with changing pressure. Formulæ have also been obtained for the optical depth τ down to level P on two assumptions, (*a*) κ (coefficient of general absorption) = constant = $\bar{\kappa}$, (*b*) $\kappa = \alpha P/(kT)^{\frac{3}{2}}$. These formulæ, both for N_r and for τ, involve g, the acceleration due to gravity, and the purpose of the present paper is to discuss the variation of N_r with g. The sign of dN_r/dg will show whether the absorption line caused by the N_r atoms should be weakened or strengthened as we pass from dwarfs to giants. If dN_r/dg is positive, N_r increases with increasing g, and the line should be weakened as we pass from dwarfs (large g) to giants (small g). To make the problem definite we keep the temperature T constant in differentiating. This is nearly, though not quite, the same thing as keeping the spectral type constant; to be exact, we keep the colour-class constant.

It is necessary to define exactly what we mean by a line increasing or decreasing in intensity. Fix attention on an arbitrary point Q of the line-contour, and let the intensity-ratio † at Q be r. Then to r corresponds a definite optical depth τ_0. We consider a series of line-contours, in different stars of differing g but constant T, and pick out for each the point Q of the line-contour where the intensity-ratio has the constant value r. Let Q_0 be the line-centre, ν_0 the frequency of Q_0, ν the frequency of Q. Then the line is said to increase or decrease in intensity according as $|\nu - \nu_0|$ increases or decreases.

If r is kept constant, τ is kept constant and equal to τ_0. This defines P (for constant T) as a function of g, and this function of g must

* *Monthly Notices*, Nov. 1928.

† The double use of r, as an intensity-ratio and as a suffix denoting stage of ionization, causes no confusion. As an intensity-ratio, r is a proper fraction; as a stage of ionization, r is an integer, 0, 1, 2, . . .

be used in the expression for N_r when differentiating with respect to g. In other words, the proper variability of P_0, the value of P at τ_0, with g, must be taken into account. An intermediate step in each investigation is therefore the determination of dP_0/dg.

The expected behaviour of lines as between dwarfs and giants has been discussed qualitatively by various writers,* in particular by Miss Payne in her monograph *Stellar Atmospheres*, chap. x. The basis of the argument has simply been that the lower pressures in the atmospheres of giants should cause increased ionization. Hence lines of the neutral atom should be weakened, and lines of the once-ionized atom should be strengthened on the low-temperature side of their maximum, weakened on the high-temperature side. In certain cases this prediction is fulfilled, but outstanding anomalies are provided by all neutral lines and especially the Balmer lines of hydrogen in cool giants, and by the lines of Sr^+. The Balmer lines of hydrogen are known to be greatly strengthened in giants as compared with dwarfs, whilst the elementary theory suggests increased ionization in the atmospheres of the giants and hence diminished abundance of the neutral hydrogen atoms which give rise to the Balmer lines. Thus the elementary theory would predict weakening of the Balmer lines in giants. Again, the principal lines of Sr^+ are observed to be always strengthened as we pass from dwarfs to giants, whilst the elementary theory predicts that they should be weakened on the high-temperature side of the maximum.

The accurate quantitative treatment of the present paper confirms these theoretical predictions *if* κ, the coefficient of general absorption, *is taken to be constant*. But pure physics indicates that the coefficient of absorption should be proportional, at constant temperature, to *the pressure of free electrons*. When we introduce the relation $\kappa \propto P$ into our calculations it is found that the sign of dN_0/dg is opposite to that given by the elementary theory and in accordance with observation; the effect arises from the increased transparency of the reversing layer with reduction of pressure, and consequent increased mass of the effective thickness of the reversing layer. I am unable to say whether the new formulæ account fully for the observed effects, but they at least give the correct sign. They should afford a useful basis of comparison with the observed march of intensities when sufficient line-contours have been measured to make the method applicable.

The method of the present paper is equally applicable to the absolute behaviour of a line and to the relative behaviour of a pair of lines. Examples of both will be given.

Lastly, if the formulæ of the present paper are found to agree with observations on line-contours they will afford a rational quantitative basis for the method of spectroscopic parallaxes, as against its present empirical basis. Study of line-contours in connexion with these formulæ should provide determinations of g for separate stars.

We now give a separate discussion of each separate problem.

2. *Problem I. Only neutral and once-ionized atoms present. Material*

* *E.g.* Stewart, *Pop. Astron.*, **31**, 88 (1923).

with a mean ionization-potential equal to that of the particular atoms considered. (*a*) κ = *constant* = $\bar{\kappa}$.—In the notation of Part I. the numbers N_0 of neutral atoms and N_1 of once-ionized atoms down to level τ_0 are given by

$$N_0 = \frac{\epsilon}{mg}\frac{P_0^2}{K_1} \quad . \quad . \quad . \quad . \quad (1)$$

$$N_1 = \frac{\epsilon}{mg}2P_0 \quad . \quad . \quad . \quad . \quad (2)$$

where P_0, the electron-pressure at level τ_0, is given by

$$\tau_0 = \frac{\bar{\kappa}}{g}P_0\left(2 + \frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad . \quad (3)$$

Keeping T and accordingly K_1 constant, and differentiating with respect to g, we have from (1), (2), and (3),

$$\frac{dN_0}{N_0} = -\frac{dg}{g} + 2\frac{dP_0}{P_0} \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{dN_1}{N_1} = -\frac{dg}{g} + \frac{dP_0}{P_0} \quad . \quad . \quad . \quad . \quad (5)$$

$$0 = -\frac{dg}{g} + \frac{dP_0}{P_0}\frac{2(P_0 + K_1)}{P_0 + 2K_1} \quad . \quad . \quad . \quad (6)$$

Substituting for dP_0 from (6) in (4) and (5) we have

$$\frac{dN_0}{N_0} = \frac{K_1}{P_0 + K_1}\frac{dg}{g} \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{dN_1}{N_1} = -\frac{P_0}{2(P_0 + K_1)}\frac{dg}{g} \quad . \quad . \quad . \quad (8)$$

We have also

$$\frac{N_0}{N_1}d\left(\frac{N_1}{N_0}\right) = -\frac{P_0 + 2K_1}{2(P_0 + K_1)}\frac{dg}{g} \quad . \quad . \quad . \quad (9)$$

These formulæ indicate that *for constant r*, as we pass from dwarfs to giants (dg negative) lines of the neutral atom should be weakened, lines of the ionized atom strengthened (these absolutely), and of course ionized lines strengthened relative to neutral lines.

3. *Problem I.* (*b*) $\kappa = \alpha P/(kT)^{\frac{3}{2}} = \alpha' P$.—Relation (3) is now replaced by

$$\tau_0 = \frac{\alpha'}{g}P_0^2\left(1 + \frac{2}{3}\frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad (10)$$

and hence differentiating with respect to g,

$$0 = -\frac{dg}{g} + \frac{dP_0}{P_0}\frac{2(P_0 + K_1)}{K_1 + \frac{2}{3}P_0} \quad . \quad . \quad . \quad (11)$$

Substituting for dP_0 in (4) and (5) we find

$$\frac{dN_0}{N_0} = -\frac{\frac{1}{3}P_0}{P_0 + K_1}\frac{dg}{g} \quad . \quad . \quad . \quad (12)$$

$$\frac{dN_1}{N_1} = -\frac{K_1 + \frac{4}{3}P_0}{2(P_0 + K_1)}\frac{dg}{g} \quad . \quad . \quad . \quad (13)$$

$$\frac{N_0}{N_1}d\left(\frac{N_1}{N_0}\right) = -\frac{K_1 + \frac{2}{3}P_0}{2(P_0 + K_1)}\frac{dg}{g} \quad . \quad . \quad . \quad (14)$$

These formulæ indicate that for constant r, as we pass from dwarfs to giants (dg negative) lines of neutral and ionized atoms should both be strengthened, and ionized lines should be strengthened relative to neutral lines.

3*a*. *Comparison with Observation.*—The following scheme summarises the results. A + sign indicates that the differential coefficient is *positive*, and hence that the corresponding lines are *weakened* as we pass from dwarfs to giants ; a − sign indicates that the corresponding lines are *strengthened*.

PROBLEM I.

Sign of :	κ = *constant.*	$\kappa = \alpha' P$.
dN_0/dg	+	−
dN_1/dg	−	−
$d(N_1/N_0)/dg$	−	−

This table should be compared with that given by Miss Payne (*Stellar Atmospheres*, p. 144), from which it differs considerably. The main point to be noted is that on the physically preferable assumption $\kappa = \alpha' P$, the neutral lines are strengthened from dwarfs to giants, whilst if κ = constant they should be weakened.

The observed facts are that for the cooler stars (for which the conditions of Problem I. are applicable) the lines of both neutral and ionized atoms increase in intensity with increasing brightness.* We may as usual take increasing brightness to mean decreasing g, and we may assume also that the same observed results are found when care is taken to keep r constant. We then see that observation is decisive in favour of the law $\kappa = \alpha' P$ as against κ = constant.

The crude qualitative argument is of course that decreasing g causes lower pressure, so increased ionization, hence weakening of the lines of the neutral atom from dwarfs to giants. The contrary result now found for the case $\kappa = \alpha' P$ is due to the way in which the level varies with changing g. Dividing equation (1) by (2) and by (10) respectively we have for κ = constant = $\bar{\kappa}$,

$$\frac{N_0}{\tau_0} = \frac{\epsilon}{m\bar{\kappa}}\frac{P_0}{P_0 + 2K_1} \quad . \quad . \quad . \quad . \quad (14')$$

* Miss Payne, *Harvard Circular*, No. 307, p. 9, 1927.

and for $\kappa = \alpha' P$ $$\frac{N_0}{\tau_0} = \frac{\epsilon}{m\alpha'} \frac{1}{\frac{2}{3}P_0 + K_1} \quad . \quad . \quad . \quad . \quad (14'')$$

From these we see at a glance that when κ = constant, N_0 decreases with decreasing P_0, whilst when $\kappa = \alpha' P$, N_0 increases with decreasing P_0.

4. *Problem II. All atoms once ionized. Second stage setting in. Mean second-stage ionization-potential equal to that of the particular atoms considered.* (*a*) κ = *constant* = $\bar{\kappa}$.—We shall find a similar result in other problems, but it is necessary to discuss each separately.

We have here

$$N_1 = \frac{2\epsilon}{mg}\left[\frac{P_0(K_2 + P_0)}{2K_2 + P_0} - K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)\right] \quad . \quad (15)$$

$$N_2 = \frac{2\epsilon}{mg}\left[\frac{\frac{1}{2}P_0 K_2}{2K_2 + P_0} + K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)\right] \quad . \quad . \quad (16)$$

$$\tau_0 = \frac{\bar{\kappa}}{g} \frac{P_0(2P_0 + 3K_2)}{P_0 + 2K_2} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

After differentiation and reduction we find

$$\frac{dN_1}{N_1} = K_2 \frac{\log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right) - \frac{\frac{1}{2}P_0}{P_0 + K_2}}{\frac{P_0(K_2 + P_0)}{2K_2 + P_0} - K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)} \frac{dg}{g} \quad . \quad (18)$$

$$\frac{dN_2}{N_2} = \frac{\frac{\frac{1}{2}P_0}{P_0 + K_2} - \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)}{\frac{\frac{1}{2}P_0}{2K_2 + P_0} + \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)} \frac{dg}{g} \quad . \quad . \quad . \quad (19)$$

$$\frac{N_1}{N_2} d\left(\frac{N_2}{N_1}\right) = \frac{\frac{2P_0K_2(2P_0 + 3K_2)}{P_0 + 2K_2}\left[\frac{\frac{1}{2}P_0}{P_0 + K_2} - \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)\right]}{\left[\frac{\frac{1}{2}P_0K_2}{P_0 + 2K_2} + K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)\right]\left[\frac{P_0(K_2 + P_0)}{P_0 + 2K_2} - K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)\right]} \frac{dg}{g} \quad (20)$$

The properties of the logarithm show that

$$\log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right) > \frac{\frac{1}{2}P_0}{P_0 + K_2}.$$

Hence

$$\frac{dN_1}{dg} > 0, \qquad \frac{dN_2}{dg} < 0, \qquad \frac{d}{dg}\left(\frac{N_2}{N_1}\right) < 0.$$

5. *Problem II.* (*b*) $\kappa = \alpha' P$.—Relation (17) is now replaced by

$$\tau_0 = \frac{2\alpha'}{g}\left[\frac{1}{2}P_0^2 - K_2^2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right) + \frac{P_0K_2^2}{P_0 + 2K_2}\right] \quad . \quad (21)$$

We now find

$$\frac{dN_1}{N_1} = \frac{\dfrac{P_0}{P_0+K_2}\left[K_2 \log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right) - \tfrac{1}{2}P_0\right]}{\dfrac{P_0(K_2+P_0)}{P_0+2K_2} - K_2 \log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)} \frac{dg}{g} \quad . \quad . \quad . \quad (22)$$

$$\frac{dN_2}{N_2} = \frac{\dfrac{1}{P_0+K_2}\left[\tfrac{1}{2}K_2 - \dfrac{P_0^2+P_0K_2+K_2^2}{P_0}\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)\right]}{\dfrac{\frac{1}{2}P_0}{P_0+2K_2} + \log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)} \frac{dg}{g} \quad (23)$$

$$\frac{N_1}{N_2}d\left(\frac{N_2}{N_1}\right) = \frac{K_2}{P_0}\frac{\left[\frac{1}{2}P_0^2 - K_2^2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right) + \frac{P_0K_2^2}{P_0+2K_2}\right]\left[\frac{\frac{1}{2}P_0}{P_0+K_2} - \log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)\right]}{\left[\frac{\frac{1}{2}P_0K_2}{P_0+2K_2} + K_2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)\right]\left[\frac{P_0(K_2+P_0)}{P_0+2K_2} - K_2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)\right]}\frac{dg}{g} \quad (24)$$

We see from the properties of the logarithm that

$$\frac{dN_1}{dg} < 0, \qquad \frac{dN_2}{dg} < 0, \qquad \frac{d}{dg}\left(\frac{N_2}{N_1}\right) < 0.$$

Hence, both for $\kappa = \text{const.} = \bar{\kappa}$ and for $\kappa = \alpha' P$ we have the same scheme of signs as in Problem I., on changing N_0 into N_1 and N_1 into N_2.

6. Problem III. The atoms in question just becoming ionized in the presence of an excess of atoms already once-ionized. (*a*) $\kappa = \text{const.} = \bar{\kappa}$. —We have here

$$N_0 = \frac{2\epsilon}{mg}\left[P_0 - K_1\log\left(1+\frac{P_0}{K_1}\right)\right] \quad . \quad . \quad (25)$$

$$N_2 = \frac{2\epsilon}{mg}K_1\log\left(1+\frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad . \quad (26)$$

$$\tau_0 = \frac{\bar{\kappa}}{g}2P_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

We find

$$\frac{dN_0}{N_0} = K_1\frac{\log\left(1+\frac{P_0}{K_1}\right) - \dfrac{P_0}{P_0+K_1}}{P_0 - K_1\log\left(1+\frac{P_0}{K_1}\right)}\frac{dg}{g} \quad . \quad . \quad . \quad (28)$$

$$\frac{dN_1}{N_1} = \frac{\dfrac{P_0}{P_0+K_1} - \log\left(1+\frac{P_0}{K_1}\right)}{\log\left(1+\frac{P_0}{K_1}\right)}\frac{dg}{g} \quad . \quad . \quad . \quad (29)$$

$$\frac{N_0}{N_1}d\left(\frac{N_1}{N_0}\right) = \frac{P_0\left[\dfrac{P_0}{P_0+K_1} - \log\left(1+\frac{P_0}{K_1}\right)\right]}{\left[\log\left(1+\frac{P_0}{K_1}\right)\right]\left[P_0 - K_1\log\left(1+\frac{P_0}{K_1}\right)\right]}\frac{dg}{g} \quad (30)$$

Hence $$\frac{dN_0}{dg} > 0, \qquad \frac{dN_1}{dg} < 0, \qquad \frac{d}{dg}\left(\frac{N_1}{N_0}\right) < 0.$$

7. *Problem III.* (*b*) $\kappa = \alpha' P$.—Relation (27) is here replaced by

$$\tau_0 = \frac{\alpha'}{g} P_0^2 \quad . \quad . \quad . \quad . \quad (31)$$

We find

$$\frac{dN_0}{N_0} = \frac{K_1 \log\left(1 + \frac{P_0}{K_1}\right) - \frac{\frac{1}{2}P_0(P_0 + 2K_1)}{P_0 + K_1}}{P_0 + K_1 \log\left(1 + \frac{P_0}{K_1}\right)} \frac{dg}{g} \quad . \quad (32)$$

$$\frac{dN_1}{N_1} = \frac{\frac{\frac{1}{2}P_0}{P_0 + K_1} - \log\left(1 + \frac{P_0}{K_1}\right)}{\log\left(1 + \frac{P_0}{K_1}\right)} \frac{dg}{g} \quad . \quad . \quad . \quad (33)$$

$$\frac{N_0}{N_1} d\left(\frac{N_1}{N_0}\right) = \frac{\frac{1}{2}P_0\left[\frac{P_0}{P_0 + K_1} - \log\left(1 + \frac{P_0}{K_1}\right)\right]}{\left[\log\left(1 + \frac{P_0}{K_1}\right)\right]\left[P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right)\right]} \frac{dg}{g} . \quad (34)$$

To discuss the sign of (32) we examine the sign of

$$\log(1 + \theta) - \frac{\theta(1 + \frac{1}{2}\theta)}{1 + \theta},$$

i.e. of

$$\log(1 + \theta) - \theta + \frac{\frac{1}{2}\theta^2}{1 + \theta},$$

which by the properties of the logarithm is negative. Hence

$$\frac{dN_0}{dg} < 0, \qquad \frac{dN_1}{dg} < 0, \qquad \frac{d}{dg}\left(\frac{N_1}{N_0}\right) < 0.$$

Again, both for $\kappa = \text{const.} = \bar{\kappa}$ and for $\kappa = \alpha' P$ we have the same scheme of signs as in Problem I.

8. *Problem IV. The atoms in question becoming ionized with ionization-potential* χ_1 *in the presence of an excess of atoms becoming ionized with ionization-potential* $\bar{\chi}_1$. (*a*) $\kappa = \text{constant} = \bar{\kappa}$.—We have here *

$$N_0 = \frac{2\epsilon}{mg} \frac{1}{\mathbf{K}_1}\left[\tfrac{1}{2}P_0^2 + (\mathbf{K}_1 - K_1)\left\{P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right)\right\}\right] \quad (35)$$

$$N_1 = \frac{2\epsilon}{mg} \frac{K_1}{\mathbf{K}_1}\left[P_0 + (\mathbf{K}_1 - K_1) \log\left(1 + \frac{P_0}{K_1}\right)\right] \quad . \quad . \quad . \quad (36)$$

$$\tau_0 = \frac{\bar{\kappa}}{g} \frac{P_0(P_0 + 2\mathbf{K}_1)}{\mathbf{K}_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

* In the following formulæ, $\mathbf{K}_1$ denotes the ionization constant K_1 calculated with $\bar{\chi}_1$ instead of χ_1. In Part I. $\mathbf{K}_1$ was denoted by $\bar{K}_1$. The change is made for convenience in printing.

We find in the usual way

$$\frac{dN_0}{N_0} = \frac{K_1}{P_0 + K_1} \frac{\frac{1}{2}P_0^2 + (\mathbf{K}_1 - K_1)\{(P_0 + K_1)\log(1 + P_0/K_1) - P_0\}}{\frac{1}{2}P_0^2 + (\mathbf{K}_1 - K_1)\{P_0 - K_1 \log(1 + P_0/K_1)\}} \frac{dg}{g} \quad (38)$$

$$\frac{dN_1}{N_1} = -\frac{1}{P_0 + K_1} \frac{\frac{1}{2}P_0^2 + (\mathbf{K}_1 - K_1)\{(P_0 + K_1)\log(1 + P_0/K_1) - P_0\}}{P_0 + (\mathbf{K}_1 - K_1)\log(1 + P_0/K_1)} \frac{dg}{g} \quad (39)$$

As a verification we note that (38) and (39) reduce to (7) and (8) when $K_1 = \mathbf{K}_1$.

To discuss the sign, consider first

$$\frac{\frac{1}{2}P_0^2}{P_0 + K_1} - K_1 \log\left(1 + \frac{P_0}{K_1}\right) + \frac{K_1 P_0}{P_0 + K_1}.$$

This is positive if it can be shown that

$$\frac{\frac{1}{2}\theta^2}{1 + \theta} > \log(1 + \theta) - \frac{\theta}{1 + \theta},$$

i.e. if

$$\int_0^\theta \frac{\theta + \frac{1}{2}\theta^2}{(1 + \theta)^2} d\theta > \int_0^\theta \frac{\theta}{(1 + \theta)^2} d\theta,$$

which is certainly true. Hence in the numerator of (38) the term independent of $\mathbf{K}_1$ is positive. But the term in $\mathbf{K}_1$ is also positive. Hence

$$\frac{dN_0}{dg} > 0,$$

and similarly

$$\frac{dN_1}{dg} < 0.$$

These results, be it noted, are independent of whether $K_1 > \mathbf{K}_1$ or $K_1 < \mathbf{K}_1$, contrary to what has sometimes been stated from the qualitative argument. Hence independent of the presence of other atoms with a different I.P., when κ = constant the lines of the neutral atom should be weakened and the lines of the ionized atom strengthened as we pass from dwarfs to giants.

9. *Problem IV.* (*b*) $\kappa = \alpha' P$.—Relation (37) is replaced by

$$\tau_0 = \frac{\alpha'}{g} P_0^2 \left(1 + \frac{2}{3}\frac{P_0}{\mathbf{K}_1}\right) \quad . \quad . \quad . \quad (40)$$

We find then

$$\frac{dN_0}{N_0} = -\frac{\frac{1}{6}\frac{P_0^3}{P_0 + K_1} + (\mathbf{K}_1 - K_1)\left[P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right) - \frac{\frac{1}{2}P_0^2}{P_0 + K_1}\right]}{\frac{1}{2}P_0^2 + (\mathbf{K}_1 - K_1)\{P_0 - K_1 \log(1 + P_0/K_1)\}} \frac{dg}{g} \quad . \quad (41)$$

$$\frac{dN_1}{N_1} = -\frac{\frac{\frac{1}{2}P_0(\frac{1}{3}P_0 + K_1)}{P_0 + K_1} + (\mathbf{K}_1 - K_1)\left[\log\left(1 + \frac{P_0}{K_1}\right) - \frac{\frac{1}{2}P_0}{P_0 + K_1}\right]}{P_0 + (\mathbf{K}_1 - K_1)\log(1 + P_0/K_1)} \frac{dg}{g} \quad . \quad . \quad (42)$$

As a verification we notice that (41) and (42) reduce to (12) and (13) when $K_1 = \mathbf{K}_1$.

To discuss the sign of the numerator of (41) consider first

$$\frac{1}{6}\frac{P_0^3}{P_0 + K_1} - K_1\left[P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right) - \frac{\frac{1}{2}P_0^2}{P_0 + K_1}\right].$$

This will be positive if it can be shown that

$$\frac{\frac{1}{6}\theta^3}{1+\theta} > \theta - \log(1+\theta) - \frac{\frac{1}{2}\theta^2}{1+\theta},$$

i.e. if

$$\int_0^\theta \frac{\frac{1}{3}\theta^3 + \frac{1}{2}\theta^2}{(1+\theta)^2} d\theta > \int_0^\theta \frac{\frac{1}{2}\theta^2}{(1+\theta)^2} d\theta,$$

which is certainly true. Hence the term in the numerator of (41) independent of $\mathbf{K}_1$ is positive. But the term in $\mathbf{K}_1$ is positive. Hence the numerator is positive and so

$$\frac{dN_0}{dg} < 0.$$

To discuss the sign of (42) consider first

$$\frac{\frac{1}{2}P_0(\frac{4}{3}P_0 + K_1)}{P_0 + K_1} - K_1\left[\log\left(1 + \frac{P_0}{K_1}\right) - \frac{\frac{1}{2}P_0}{P_0 + K_1}\right].$$

This will be positive if it can be shown that

$$\frac{\frac{1}{2}\theta(\frac{1}{3}\theta + 1)}{\theta + 1} > \log(1+\theta) - \frac{\frac{1}{2}\theta}{1+\theta},$$

i.e. if

$$\int_0^\theta \frac{\frac{2}{3}\theta^2 + \frac{4}{3}\theta + \frac{1}{2}}{(1+\theta)^2} d\theta > \int_0^\theta \frac{\theta + \frac{1}{2}}{(1+\theta)^2} d\theta,$$

which is certainly true. Hence by a similar argument to the above

$$\frac{dN_1}{dg} < 0.$$

These results are independent of whether $K_1 < \mathbf{K}_1$ or $K_1 > \mathbf{K}_1$. Hence independent of the presence of other atoms with a different I.P., when $\kappa = \alpha' P$ the lines of both neutral and once-ionized atoms should be strengthened as we pass from dwarfs to giants.

Similar calculations show that both in case (*a*) and in case (*b*)

$$\frac{d}{dg}\left(\frac{N_1}{N_0}\right) < 0,$$

so that lines of ionized atoms are strengthened relative to those of neutral atoms as we pass from dwarfs to giants.

The scheme of signs is thus the same as in Problem I., of which of course Problem IV. is a generalisation. We notice throughout a

marked difference in the behaviour of the neutral lines as between case (*a*) and case (*b*). If we take κ = constant, the neutral lines are weakened as we pass from dwarfs to giants; if we take $\kappa = \alpha' P$, they are strengthened.

10. *Application of the results of Problem IV. to Hydrogen in the cooler Stars.*—The foregoing results should be applicable to all lines in the cooler stars, where second-stage ionization is not appreciable. But the most interesting application is to the Balmer lines of hydrogen. It follows from the analysis that when κ = constant the Balmer lines (being due to the neutral atom) should be weakened from dwarfs to giants, but when $\kappa = \alpha' P$ they should be strengthened. The observed fact is that the Balmer lines are strengthened. Now the law κ = constant is a mere assumption, made for convenience, whilst the law $\kappa = \alpha' P$ is that predicted from pure physics. The strengthening of the Balmer lines from dwarfs to giants, which has hitherto been not only unexplained but in opposition to the elementary theory, now follows naturally from the theory now expounded when the proper behaviour of κ is taken into account.

In the cooler stars, χ_1 for hydrogen being so much larger than the I.P. of the atoms with which it is surrounded, K_1 is small compared with $\mathbf{K}_1$. It will be noticed that in (38) K_1 occurs as a factor in the expression for dN_0/dg. Thus when $\bar{\kappa}$ = constant, dN_0/dg is very small for hydrogen. The physical reason for this is that almost all of the hydrogen atoms are in the neutral state, and reduction of pressure consequent on reduction of g causes an inappreciable diminution in the number of neutral atoms, χ_1 being so large. But when $\kappa = \alpha' P$ reduction of g involves a different change in P, owing to change of level.

We see in fact from (41), on neglecting K_1 compared with $\mathbf{K}_1$ and with P_0, that when $\kappa = \alpha' P$ the value of dN_0/dg is approximately given by

$$\frac{dN_0}{N_0} = -\frac{\frac{1}{6}P_0^3 + \frac{1}{2}P_0\mathbf{K}_1}{\frac{1}{2}P_0^2 + P_0\mathbf{K}_1}\frac{dg}{g} = -\frac{\frac{1}{2}(1 + \frac{1}{3}P_0/\mathbf{K}_1)}{1 + \frac{1}{2}P_0/\mathbf{K}_1}\frac{dg}{g} \quad . \quad . \quad . \quad (43)$$

Now P_0 is of the order of $\mathbf{K}_1$ for the cooler stars, $\mathbf{K}_1$ being calculated for a $\bar{\chi}_1$ corresponding to a low-temperature maximum. Putting $P_0 = \mathbf{K}_1$ in (43), we see that the order of magnitude of dN_0/dg is given by *

$$\frac{dN_0}{dg} = -\frac{4}{9}\frac{dg}{g}.$$

Hence approximately

$$N_0 \propto g^{-\frac{4}{9}}. \quad . \quad . \quad . \quad . \quad (44)$$

This law of variation of N_0 should be sufficient to cause a conspicuous difference between dwarfs and giants, but I have access to no data which would test it quantitatively.

* It would be more satisfactory to calculate a table of values of N_0 for different g, from (35) and (40).

It is easy to show that it follows from the theory that the Balmer lines should be much more sensitive to variation of g in the cooler stars than in the hotter stars. Near the maximum of the Balmer lines the conditions of Problem III. obtain. P_0 is of the order of K_1. Putting $P_0 = K_1$ in formula (32) we see that when $\kappa = \alpha' P$ the variation of N_0 with g near the maximum is given by

$$\frac{dN_0}{N_0} = \frac{\log 2 - \frac{3}{4}}{1 - \log 2}\frac{dg}{g} = -\frac{0{\cdot}05685}{0{\cdot}30685}\frac{dg}{g}$$
$$= -0{\cdot}185 dg/g.$$

Hence near the maximum

$$N_0 \propto g^{-0{\cdot}185} \quad . \quad . \quad . \quad . \quad (44')$$

which is a much slower variation than (44).

Another point to notice is that the Balmer lines will be more sensitive to variation of g than the lines of neutral atoms of low ionization-potential in the cooler stars. For these atoms the conditions are those of Problem I., and for $\kappa = \alpha' P$ the behaviour is given by formula (12). Putting $P_0 = K_1$ to see the order of magnitude we have for these atoms

$$\frac{dN_0}{N_0} = -\frac{1}{6}\frac{dg}{g}$$

or

$$N_0 \propto g^{-\frac{1}{6}} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Comparison with (44) shows immediately the increased sensitivity of the Balmer lines. The Balmer lines should be less sensitive than the lines of ionized atoms of low I.P., for formula (13) with $P_0 = K_1$ gives $N_1 \propto g^{-7/12}$.

It is known that the Balmer lines are very sensitive to increase of luminosity in the cooler stars, and it will be interesting to know whether the above predictions are confirmed by observation.

10*a*. *Alternative Treatment for Problem IV.*—The above results are so striking that it seems worth while giving a slightly different mode of derivation.

Since K_1 is small compared with $\mathbf{K_1}$, the value of N_0 given by (35) is little different from

$$N_0 = \frac{2\epsilon}{mg}\left[P_0 + \frac{\frac{1}{2}P_0^2}{\mathbf{K_1}}\right]. \quad . \quad . \quad . \quad (46)$$

The absence of K_1 (the ionization constant for hydrogen) from this expression shows that the ionization of hydrogen is of little importance at low temperatures; the number of effective hydrogen atoms practically depends only on the abundance ϵ of hydrogen and on P_0, the depth to which we see.

Take now case (*a*). Then τ_0 is given by (37), and so from (46)

$$N_0 = \epsilon\tau_0/m\bar{\kappa} \quad . \quad . \quad . \quad . \quad (47)$$

which is independent of g; dN_0/dg is approximately zero—we see to

the same depth of hydrogen whatever is g. This is in agreement with what we saw from (38).

Next take case (*b*). Here, τ_0 is given by (40), and so dividing by (46) we have

$$N_0 = \frac{2\epsilon\tau_0}{m\alpha'} \frac{1 + \frac{1}{2}P_0/K_1}{P_0(1 + \frac{2}{3}P_0/K_1)} \quad . \quad . \quad . \quad (48)$$

When g decreases, P_0 decreases, and hence here N_0 increases, as before.

11. *Physical Explanation.*—It would be desirable to give a direct physical explanation of the common result of the discussion of Problems I.–IV., namely, that κ = constant gives weakening of neutral lines from dwarfs to giants, whilst $\kappa = \alpha' P$ gives strengthening. Unfortunately, all we can say without calculation is that there are two opposing tendencies at work in case (*b*), whilst the tendency is only one way in case (*a*).

When κ = constant, the total mass μ per unit area down to level τ_0 remains constant. Reduction of g causes reduction of pressure, hence increased ionization and reduced strength of neutral lines. But when $\kappa = \alpha' P$, the mass μ does not remain constant; it increases. Although the number of free electrons per unit mass is increased by decreasing g, the electrons are much further apart, and the net result is found to be that the matter is more transparent. Hence, for given total opacity (optical depth), the mass μ is increased. In other words, the "mass of material above the photosphere" is increased. This more than counterbalances the decrease in the *proportion* of neutral atoms due to increased ionization, and the lines due to neutral atoms are now increased in strength. Increased thickness of reversing layer wins the day over increased ionization. The possibility of this kind of effect occurring was pointed out by Stewart* and re-stated by Miss Payne in her monograph on stellar atmospheres.

The increase in the mass of the reversing layer with decreasing g, when $\kappa \propto P$, is readily demonstrated. Take Problem I. as typical. Since $\mu = p/g$, where p is the total pressure, we find by reference to Part I., equation (22), that

$$\mu = \frac{P_0}{g}\left(2 + \frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad . \quad (48')$$

Differentiating with regard to g and using the value of dP_0/dg given by (11), § 3, we find

$$\frac{d\mu}{\mu} = -\frac{K_1 + \frac{1}{3}P_0}{2K_1 + P_0}\frac{dg}{g}.$$

Thus $d\mu/dg$ is negative, or μ increases with decreasing g. (When κ = constant = $\bar{\kappa}$, $\mu = \tau_0/\bar{\kappa}$ and so is constant.) Again, dividing (48′) by (10) we have

$$\mu = \frac{\tau_0}{\alpha'}\frac{2K_1 + P_0}{P_0(K_1 + \frac{2}{3}P_0)},$$

which shows at once that μ increases as P_0 decreases.

* *Pop. Astron.*, **33**, 303, 1925.

It does not appear possible to demonstrate the dominance of increased mass over increased ionization without a separate calculation for each problem, as given above in §§ 2–9.

12. *Problem V. Three successive stages of ionization present. Material with mean first- and second-stage I.P.'s equal to those of the particular atoms considered.* (*a*) κ = *const.* = $\bar{\kappa}$.—This is the problem of the behaviour of the principal lines of the ionized atom. We have here

$$N_0 = \frac{\epsilon}{mg}\frac{P_0^3}{K_1(P_0+2K_2)} \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$N_1 = \frac{2\epsilon}{mg}\left[\frac{P_0(P_0+K_2)}{P_0+2K_2} - K_2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)\right] \quad . \quad (50)$$

$$N_2 = \frac{2\epsilon}{mg}\left[\frac{\frac{1}{2}P_0K_2}{P_0+2K_2} + K_2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)\right] \quad . \quad . \quad (51)$$

$$\tau_0 = \frac{\bar{\kappa}}{g}\frac{P_0(3K_2+2P_0+P_0^2/K_1)}{2K_2+P_0} \quad . \quad . \quad . \quad . \quad (52)$$

We are interested chiefly in the lines of the once-ionized atom, so we consider first N_1. We find in the usual way from (50) and (52), after some algebra,

$$\frac{dN_1}{N_1} = \frac{K_2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right) - \frac{\frac{1}{2}P_0(K_2+P_0^2/K_1)}{K_2+P_0+P_0^2/K_1}}{\frac{P_0(P_0+K_2)}{P_0+2K_2} - K_2\log\left(1+\frac{1}{2}\frac{P_0}{K_2}\right)}\frac{dg}{g} \quad . \quad (53)$$

This reduces to (8), as it should, on putting $K_2 = 0$.

To discuss the sign of (53) we notice that the numerator of the coefficient of dg/g can be written in the form

$$\tfrac{1}{2}K_2^2\int_0^{P_0}\left(3+2\frac{P}{K_2}+\frac{P^2}{K_1K_2}\right)\left(1-\frac{P^2}{K_1K_2}\right)\frac{P\,dP}{(P+2K_2)(K_2+P+P^2(K_1)}.$$

This formula is easily verified by direct differentiation. The integral just written is positive for $P_0 < K_1^{\frac{1}{2}}K_2^{\frac{1}{2}}$ and for some range of larger values of P_0.

At the theoretical maximum of the lines P_0^2 is somewhat greater than K_1K_2 (see Part I.) though of much the same order. Hence here dN_1/dg is probably small. If P_0 were independent of T, at temperatures below T_{max} P_0 would exceed $K_1^{\frac{1}{2}}K_2^{\frac{1}{2}}$, and at temperatures above T_{max} P_0 would be less than $K_1^{\frac{1}{2}}K_2^{\frac{1}{2}}$, approximately, K_1 and K_2 being increasing functions of T. Thus were P_0 independent of T, dN_1/dg would be positive for $T > T_{max}$ and negative for T considerably smaller than T_{max}. Hence on the assumption κ = constant, if P_0 were independent of T the lines of the ionized atom would be weakened from dwarfs to giants for $T > T_{max}$, strengthened for T considerably smaller than T_{max}. This is in agreement with the qualitative treatment of

the elementary theory. Actually P_0 is a complicated function of T given by (52), from which it may be deduced that

$$dP_0/dT > 0.$$

A detailed investigation does not seem worth while, but we may conclude that the value of dN_1/dg as given by (52) is somewhat smaller than we should infer if we assumed P_0 independent of T. The behaviour of dN_1/dg could of course easily be found numerically in any given case from (53).

13. *Problem V.* (*b*) $\kappa = \alpha' P$.—We have here approximately

$$\tau_0 = \frac{\alpha'}{g} P_0^2 \left(1 + \frac{2}{3}\frac{P_0}{K_1}\right) \quad . \quad . \quad . \quad (54)$$

Combining this with (50) we find in the usual way,

$$\frac{dN_1}{N_1} = \frac{K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right) - \frac{P_0}{P_0 + 2K_2}\left[\frac{\frac{1}{2}P_0^2 + \frac{3}{2}P_0K_2 + 2K_2^2}{P_0 + 2K_2} + \frac{\frac{1}{5}P_0^2(P_0 + 3K_2)}{(P_0 + K_1)(P_0 + 2K_2)}\right]}{\frac{P_0(P_0 + K_2)}{P_0 + 2K_2} - K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right)} \frac{dg}{g} \quad (55)$$

This reduces to (13), as it should, on putting $K_2 = 0$.

The coefficient of dg/g is certainly negative if it can be shown that

$$K_2 \log\left(1 + \frac{1}{2}\frac{P_0}{K_2}\right) - \frac{P_0(\frac{1}{2}P_0^2 + \frac{3}{2}P_0K_2 + 2K_2^2)}{(P_0 + 2K_2)^2} < 0 \quad . \quad (56)$$

To investigate the sign of this expression, write

$$\tfrac{1}{2}P_0/K_2 = \theta.$$

It becomes, on dividing out by K_2,

$$\log(1 + \theta) - \frac{\theta(\theta^2 + \frac{3}{2}\theta + 1)}{(\theta + 1)^2}$$

$$= \frac{\frac{1}{2}\theta^2}{(1 + \theta)^2} - [\theta - \log(1 + \theta)]$$

$$= \frac{\frac{1}{2}\theta^2}{(1 + \theta)^2} - \int_0^\theta \frac{t\,dt}{1 + t},$$

which is certainly negative. Hence inequality (56) is established and so

$$\frac{dN_1}{dg} < 0.$$

Hence N_1 increases when g decreases, *i.e.* the lines of the ionized atom are always strengthened as we pass from dwarfs to giants. Thus in case (*b*), $\kappa = \alpha' P$, we have a unique result, the same for $T > T_{max}$ and for $T < T_{max}$.

14. *Comparison with Observation.*—Observation shows that the lines of Sr^+ are always strengthened from dwarfs to giants. Miss Payne

writes: "The strengthening of Sr^+ with high luminosity is one of the best-attested facts of observational astrophysics, and it is a serious deficiency in theory if the observed behaviour of the lines in the hotter stars cannot be explained." * Miss Payne is referring to the qualitative argument that for $T > T_{max}$ the lines should be weakened from dwarfs to giants, which, as we have seen, is substantially the prediction of our quantitative method for κ = constant. But the observed behaviour is fully explained when we take κ proportional to electron-pressure. On assumption (*b*), $\kappa = \alpha' P$, we have found that the lines of the ionized atom should always be strengthened from dwarfs to giants, as observed.

15. *Approximate formula in Problem V., case* (*b*).—The exact formula is (55). In this problem K_2 is always small compared with P_0. Neglecting K_2 compared with P_0 we have approximately, from (55),

$$\frac{dN_1}{N_1} = -\left[\tfrac{1}{2} + \frac{\tfrac{1}{6}P_0}{P_0 + K_1}\right]\frac{dg}{g} \quad . \quad . \quad . \quad (57)$$

Hence N_1 varies slightly more rapidly with g than $g^{-\frac{1}{2}}$. This rapid change of N_1 with g accounts for the substantial change of intensity of the Sr^+ lines from dwarfs to giants.

16. *Problem V. Behaviour of lines of neutral atom.* (*a*) κ = *constant* = $\bar{\kappa}$.—We differentiate (49) with respect to g, P_0 being a function of g given by (52). We find in the usual way

$$\frac{dN_0}{N_0} = \frac{2K_2 + P_0}{K_2 + P_0 + P_0^2/K_1}\frac{dg}{g} \quad . \quad . \quad . \quad (58)$$

Hence the lines of the neutral atom should be weakened as we pass from dwarfs to giants. This result is merely an extension of the result of (7), § 2, when second-stage ionization is allowed for. Equation (58) reduces in fact to (7) on putting $K_2 = 0$.

(*b*) $\kappa = \alpha' P$.—Differentiating (49) with respect to g and using (54), we find

$$\frac{dN_0}{N_0} = \frac{-\tfrac{1}{3}P_0^2 + K_1K_2}{(P_0 + 2K_2)(P_0 + K_1)}\frac{dg}{g} \quad . \quad . \quad . \quad (59)$$

This is the extension of (12), to which it reduces when K_2 is put equal to zero. Clearly $dN_0/dg < 0$ at low temperatures, but it will be positive or only slightly negative on the high-temperature side of the maximum.

17. *The Theory of "Spectroscopic Parallaxes."*—The formulæ of the present paper are in a form suitable for comparison with observations on line-contours for stars of the same temperature but different g. The recent work of Dunham,† of Miss Payne,‡ and of Payne and Hogg § on the contours of hydrogen and calcium lines gives precisely the data required, but it would be out of place to attempt to analyse them in the present paper, which is concerned with stating the theory. Perhaps

* *Stellar Atmospheres*, p. 148.
† *Harvard Bulletin*, No. 858, p. 7.
‡ *Ibid.*, p. 10.
§ *Harvard Circular*, No. 334.

the present author may be permitted to express the wish that the work of analysis will be undertaken by the workers who have made the actual observations, as they are best in a position to estimate the accuracy of the data for the purpose in hand.

In the meantime it may be pointed out that the method and formulæ of the present papers (Parts I. and II.) afford the basis for a rational treatment of the method of spectroscopic parallaxes, as opposed to the present empirical basis. The rational treatment would be as follows: The constant α' can be determined, as shown in Part I., by the method of maxima. Hence for a given point r on a line-contour P_0 is known as a function of g, from the equation for τ_0 for the appropriate problem. The formulæ of the present paper show how to calculate $d(\log N)/d(\log g)$ as a function of P_0 (at constant temperature, and hence in principle as a function of g). Measurement of line-contours and reduction by Unsöld's method provide the actual value $\Delta(\log N)$ as between two stars—assuming the composition ϵ to be the same for the two, as tacitly assumed throughout. Hence if g be known for one of the stars, and $d(\log N)/d(\log g)$ be calculated from the formula, $\Delta(\log g)$ may be found, and so g determined for the second star.

There would be no need to determine $d(\log N)/d(\log g)$ explicitly as a function of g. The actual procedure would probably be as follows: Take first the appropriate formulæ of the type

$$\tau_0 = \frac{\alpha}{g}\phi(P_0, T),$$

$$\frac{d(\log N)}{d(\log g)} = \psi(P_0, T),$$

as found in this and the preceding paper. Take a star of known g, and from the known value of α (astrophysical value from maxima) calculate P_0 for the temperature in question, from the first of the above formulæ. Introducing this in the second, calculate $d(\log N)/d(\log g)$. Determine $\Delta(\log N)$ from observation. Calculate $\Delta \log g$ from the formula

$$\Delta \log g = \frac{d(\log g)}{d(\log N)}\Delta \log N.$$

Hence determine g for the second star.

An important point to notice is that if Unsöld's method be followed in detail, the value derived for $\Delta \log N$ will be independent of the constants in his formula. For since we are working with constant r, by Schuster's formula we have for the two stars,

$$\frac{1}{1 + N\alpha_\nu} = \frac{1}{1 + (N + \Delta N)(\alpha_\nu + \Delta\alpha_\nu)},$$

where $\alpha\nu$ is the atomic absorption coefficient. Hence

$$\alpha_\nu \Delta N + N\Delta\alpha_\nu = 0.$$

But according to Unsöld a_ν is of the type

$$a_\nu = \frac{a}{(\nu - \nu_0)^2},$$

whence

$$\frac{\Delta a_\nu}{a_\nu} = -\frac{2\Delta\nu}{\nu - \nu_0},$$

so that

$$\Delta (\log N) = \frac{2\Delta\nu}{\nu - \nu_0}.$$

Here $2\Delta\nu$ is the increase in the width of the line, from the first star to the second, for constant intensity ratio r. Thus $\Delta(\log N)$ is computed directly from $\Delta\nu$.

There seems no reason in principle why a complete table of values of g for all stars should not be constructed in this way, by spectrophotometric observations of line-contours and comparison with stars of known g. The formulæ would, of course, need a thorough preliminary test on stars of known g.

It must be borne in mind that, as expressly mentioned in Part I., no allowance has been made for selective radiation-pressure. The values of g required in the present paper require in principle to be corrected for selective radiation-pressure. A separate investigation is required to decide how far this is appreciable. It is probable, however, as indicated by chromospheric theory, that the chief effect of selective radiation-pressure is to deepen the cores of lines. The wings of lines correspond to deeper levels (larger τ_0), and here the selective effect of radiation-pressure will be smaller. Hence the formulæ of the present paper will be expected to be valid provided we do not go too close to the centre of the line.

If M is the mass of a star, L its bolometric luminosity, R its radius, we have $g \propto M/R^2$, $L \propto R^2T^4$, whence $g \propto MT^4/L$. Thus a knowledge of g requires a knowledge of mass and parallax; hence the formulæ of the present paper could best be tested on double stars for which mass and parallax can both be found. For stars whose mass is unknown, but whose parallax is known, the formula $M \propto gL/T^4$ shows that a spectrophotometric determination of g will determine the mass. We have thus a method of determining the masses of individual stars by a combination of parallax and spectrophotometric observations. The ideas of this paragraph are not original, they were first clearly enunciated by Pannekoek * in 1922. It seems, however, worth while restating them now that accurate formulæ are presented.

If neither M nor L are known, a spectrophotometric determination of g determines the ratio L/M. For a group of stars for which a mean M can be assumed, this determines the mean L, and so the mean parallax.

Our formulæ suggest other methods of determining g, without even

* *B.A.N.*, No. 19, 1922. See also the same author's work in *Observatory*, **46**, 304, 1923.

a standard star of known g being required. It has been shown in Part I. how the composition ϵ may be determined from line-contours, for stars of known g. Suppose this is found for a group of stars, and assumed to be the same for other stars. Observation of the ratio N_1/N_0 for a pair of lines in the same star determines P_0, and then inserting this in the appropriate formula and using the absolute value of N_0 or N_1 and the value of ϵ, we obtain g directly. This method involves, however, an accurate knowledge of the absolute value of the constant in the atomic absorption coefficient, whereas the differential method is independent of this.

It seems worth while emphasizing that the constancy of τ_0 is fundamental in the analysis of this paper, and τ_0 is only constant when r is constant. Observations of line-width for constant r require the determination of a line-contour. Hence determinations of "intensities" of stellar absorption lines on arbitrary scales are of no value in this connection.

When r is kept constant, the intrinsic atomic opacity (selective) is not required, and has nowhere been used. If, however, r is not kept constant, the atomic absorption coefficient would be required. For example, if we made observations on the line-centre, as the residual intensity changed τ_0 would change, and the atomic absorption coefficient would appear in the calculations.

18. *Summary.*—(1) Formulæ are obtained which determine the variation with g, at constant T, of the number of atoms responsible for the formation of a given absorption line.

(2) The behaviour depends on whether κ, the coefficient of general absorption, is constant throughout the stellar atmosphere or is proportional at each point to the electron-pressure P.

(3) When κ = constant, the lines of the neutral atom should decrease in intensity from dwarfs to giants, but when $\kappa \propto P$ they should increase. The physical explanation of this is that when κ = constant the mass of the reversing layer remains constant, and so reduction of pressure, causing increased ionization, leads to a reduction in the number of neutral atoms. But when $\kappa \propto P$ the mass of the reversing layer increases with decreasing pressure, and the increased mass of the reversing layer more than counterbalances the reduction in the proportion of neutral atoms due to increased ionization. The increase in mass of the reversing layer when $\kappa \propto P$ is a consequence of the increased transparency which results from decreased pressure.

(4) Lines of ionized atoms should be strengthened from dwarfs to giants on both hypotheses, but more so if $\kappa \propto P$ than if κ = constant. On each hypothesis lines of ionized atoms should be strengthened relative to lines of neutral atoms.

(5) The observed fact that lines of both neutral and ionized atoms increase in intensity from dwarfs to giants decides in favour of $\kappa \propto P$ and against κ = constant. The law $\kappa \propto P$ is that given by pure physics when the opacity arises from free electrons, and this hypothesis therefore receives observational support.

(6) In particular, the observed strengthening of the Balmer lines

from dwarfs to giants receives theoretical explanation. The number of effective atoms should vary approximately as $g^{-4/9}$ for the cooler stars, but as a much smaller (negative) power of g for the hotter stars, when $\kappa \propto P$. This is in agreement with the observed sensitivity of the Balmer lines to absolute luminosity in the cooler stars. The theory predicts further that the Balmer lines should be more sensitive than the lines of neutral atoms of low I.P.

(7) The physical explanation in the case of hydrogen is that the I.P. is so high that reduction of pressure causes a negligible increase in ionization at low temperatures, consequently the increased thickness of the reversing layer consequent on reduced pressure if $\kappa \propto P$ has its full effect. At higher temperatures the increase in ionization becomes appreciable and the strengthening of the Balmer lines with reduced pressure is diminished.

(8) When $\kappa \propto P$, the lines of the ionized atom such as Sr^+ should increase in intensity with increasing luminosity, on *both* sides of the temperature maximum. This is in agreement with observation.

(9) Comparative spectrophotometric observations on the same line in two stars of the same temperature should permit the determination of g for the second star if g for the first star is known. A rational basis for the method of spectroscopic parallaxes is sketched. Spectrophotometric observations on stars of known parallax should determine the masses of individual stars, as first pointed out by Pannekoek. Other methods of determining g from a knowledge of the atomic absorption coefficient are sketched.

(10) Comparative spectrophotometric observations must be made on line-contours at points of constant intensity-ratio in order that the methods of the present paper shall be applicable.

High Dispersion Prism Spectra. By J. Evershed, F.R.S.

In designing a spectrograph and a spectroheliograph for my observatory at Ewhurst five years ago, I was faced by a very serious difficulty in providing for sufficient dispersive power for the work I intended to undertake. I had available two 45° prisms of 6-inch aperture, which I had used successfully in Kashmir for photographing prominences and calcium flocculi in the H or K lines. For this purpose sufficient dispersion was obtained by single transmission through the prisms and a camera lens of 7 feet focal length. In my new observatory I intended to photograph the sun in the hydrogen line α, which requires much greater dispersion in the red, and in addition I wished to continue the study of line-of-sight motions in prominences and sunspots which was begun at Kodaikanal. To make any progress in this kind of work very great dispersive power is essential.

It seemed doubtful whether sufficient dispersion could be obtained

with the prisms, especially for work at the red end of the spectrum, and the obvious course was to procure a grating, such as I had been accustomed to use in Kodaikanal. Large gratings are, however, very difficult to obtain at the present time, and negotiations proving unavailing, I was compelled to make the attempt with prisms.

In order to get a reasonably high dispersion, I arranged the prisms in an auto-collimating Littrow spectrograph with a plane mirror to return the light, giving the equivalent of four prisms. The result was disappointing, for I found the definition of spectrum lines, although good enough with single transmission and a 7-foot camera, failed entirely with double transmission and a 16-foot camera, owing to the want of homogeneity in the large blocks of glass.

The next step was to have the prisms reannealed and refigured, an expensive and rather risky proceeding, which might or might not give good results. Fortunately the reannealing, in the hands of the Parsons Optical Company, and the figuring by Adam Hilger Ltd. was entirely successful. On receipt of the perfected prisms, in June 1925, I was able to complete the construction of the spectroheliograph. The definition of spectrum lines then appeared perfect, and spectroheliograph work was started in the autumn of that year. I was able to photograph prominences at the limb, and markings on the disc in Hα light. In the H and K region the dispersion was sufficient for line shift work in the prominences.

The dispersive power between H and K was 0·8 mm. to the angstrom unit, and at Hα about 1 mm. to 8 angstroms. This dispersion in the red is too small for the best results with the spectroheliograph for hydrogen images, as a very narrow second slit has to be used to isolate the central part of the red hydrogen line. In practice, the second slit should not be much narrower than $\frac{1}{10}$th of a millimeter, which is about the width of the hydrogen line on this scale.

It appeared that a great improvement would result if I could replace the silvered mirror at the back of the prisms by a 30° reflecting prism, of equal aperture, which would add the equivalent of a 60° prism to the dispersing train, and secure the advantage of an internal reflecting surface. I was most fortunate at this stage to receive a visit from Professor Turner, who strongly advised an application to the Government Grant Committee of the Royal Society for funds for the construction of a large reflecting prism. The trouble was to produce a large block of glass of the necessary homogeneity, and after a year or so of ineffectual attempts this appeared a rather hopeless undertaking. I consider that it is entirely due to Professor Turner's continued interest and helpful influence that finally, this year, a beautiful piece of glass has been produced by the Parsons Co., and worked to perfection by Messrs. Hilger.

The prism was installed in my spectroheliograph in September, and has proved a great success. The hydrogen line is now considerably wider than the second slit, and the definition of spectrum lines is as perfect as could be wished.

Using the instrument as a spectrograph, the angular dispersion in

the H and K region now exceeds that given by the third order of a grating of 15,000 lines to the inch, so that if I had a large grating in the place of the prisms, and the same 16-foot lens, I should get nothing better, nor so good, as grating spectra in the higher orders give a certain amount of scattered light, to say nothing of ghosts. The prism spectrum gives beautiful contrast and perfectly defined emission lines of arc spectra. Of course the prisms cannot compete with a grating for work in the visible part of the spectrum towards the red end.

The actual linear dispersion between H and K is just over 1 mm. to the angstrom unit, but it falls off rapidly towards the blue, so that at λ 4500 it is about $\frac{1}{2}$ mm. to the angstrom. For studying line-of-sight movements in sunspots, in the region between Hβ and Hγ, at least double this dispersion is necessary, and for this work I have for long been experimenting with a hollow prism containing ethyl cinnamate.

This liquid has a remarkably high dispersive power. I find the refractive index is equal to that of the glass walls of the prism for green light, but greatly exceeds ordinary glass in the violet. As the liquid is non-volatile, it is not difficult to maintain a uniform temperature, even when the surface is exposed to the air. I find that it is possible to maintain absolutely uniform refracting power throughout the prism for an hour or two. This condition of uniformity has never been secured in large blocks of glass, especially the denser kinds which give comparable dispersion. In using the prism it is necessary to employ a small stirrer worked by an electric motor, to mix the liquid occasionally.

Until recently I have had only partial success with the cinnamate prism. It has, in fact, entailed a large amount of wasted effort, ending with a bad but not fatal accident to the prism, which might well have put an end to further experiments, but actually gave me a clue which has led to final success.

The most serious difficulty was the distortion of the surfaces of the prism, due to the action of the cement; and another trouble was the short time after stirring the liquid that definition remained good.

After many experiments I have now, I think, completely overcome these defects.

A new prism was made by Messrs. Hilger, in which the sides are worked to perfect planes, and the edges of the plates, and the other parts of the prism, are carefully worked true and polished. By a clever manipulation the prism was assembled and fitted together without cement, the parts being pressed into optical contact. This prism gave perfect definition, but failed after a few weeks, from the fact that the cinnamate slowly made its way between the parts in optical contact, and finally the prism leaked. I then took the plates apart, cleaned them, and reassembled the prism, using a cement that does not require strong heating. This up to the present has proved quite successful, and the prism is now permanently installed in an underground chamber, ready for high dispersion work. It is only necessary to run the stirrer for one minute, after which it may be left for an hour or two before a second stirring becomes necessary.

The effective aperture of the prism is nearly 4 inches, and it is backed by a 30° glass reflecting prism in an auto-collimating spectrograph of about 16 feet focal length. Spot spectra taken with this instrument are comparable with the best grating spectra. The scale is 1 mm. to the angstrom at λ 4500, and the definition of the lines is perfect. In the H and K region the scale is 2 mm. to the angstrom, but the exposure time here is four minutes, instead of about as many seconds, which is all that is required at λ 4500. I might mention that commercial cinnamate is usually slightly coloured, and is very absorptive for the H and K region. To get over this difficulty, a chemical friend, Mr. Saville of the Royal Institution, very kindly redistilled my cinnamate at an extremely low pressure, producing a perfectly colourless liquid. This reduces exposure times to a minimum. I hope eventually to use both cinnamate prisms in train together. This would give over 2 mm. per angstrom at Hγ, and $1\frac{1}{3}$ mm. at Hβ.

From the experience I have gained in working with these prisms, I am hopeful that a much larger prism could be made, and successfully used in a spectroheliograph or spectrograph.

1928 *November* 30.

The Excitation of Line and Band Spectra in Comets by Sunlight.

By H. Zanstra, Ph.D.

(*Communicated by S. Chapman.*)

1. *General Properties of Comets.*

As is well known, the spectrum of a comet's head presents the following general features. At a great distance from the sun it is continuous, but, as the comet approaches the sun, a spectrum of bright bands appears also, the essential feature of which is the Swan spectrum,* and, in some cases, also the Cyanogen bands. If the comet has a small perihelion distance, as the Comet Wells of 1882, the bright D lines of sodium may become predominant in its spectrum when near perihelion ; these lines always appear later than the Swan bands. In the Great Comet of 1882, with a perihelion distance of only 0·008 astronomical units, several bright lines attributed to iron were observed also near perihelion.

During the time when the band and line spectra are emitted, the head of the comet usually appears as a nebulous mass, or coma, inside of which is a brighter nucleus of much smaller dimensions. The spectrum of bright bands and lines of the head is always accompanied by a continuous spectrum in which Fraunhofer lines have been identified in many cases, so that it is most likely due to reflected sunlight; this continuous spectrum is strongest for the nucleus, but it occurs

* In older literature often referred to as "hydrocarbon spectrum."

also in the coma. If the head of a comet shows a strong bright line and band spectrum, the comet's tail is usually well developed. When the tail is absent, the spectrum of the head, as a rule, is practically continuous.

Passing from the head of a comet towards the tail, the character of the spectrum gradually changes into that of a bright band spectrum of ionised molecules. The characteristic bands in the tail spectrum were identified by Fowler with his low-pressure spectrum of carbon monoxide * which, according to recent work, must be attributed to singly ionised carbon monoxide, CO^+. The tail of the Comet Morehouse showed also the kathode nitrogen bands,† which are now known to be due to singly charged nitrogen, N_2^+.

Though the previous outline characterises the general behaviour of comets as regards their spectrum, it should be borne in mind that the individual cases show a wealth of detail, and often a peculiar and unexpected behaviour, which it is hard to bring under a general scheme.

It is fairly generally agreed that a comet consists of meteorites, or at any rate solid bodies, which contain occluded gases or substances which may evaporate, a view which was put forward by Vogel. If such a collection of meteorites is far away from the sun, it gives a continuous spectrum due to reflected sunlight, as is observed for the comets. But as they approach the sun in their paths under gravity, the occluded gases will be freed by the sun's heat, and the coma develops. Still nearer to the sun, solid compounds of sodium, and subsequently iron or its compounds, might also be evaporated. Vogel, in support of his view, investigated the spectrum of the occluded gases freed from meteorites found on the earth, produced in a discharge tube, and found the Swan bands together with the Ångstrom and Thalen carbon monoxide bands, the latter of which, however, do not occur in comets.

The conclusion that the nucleus of a comet consists of solid or liquid bodies of an appreciable size seems unavoidable, since the acceleration of the nucleus in its orbit is very generally that of gravity.‡ In Russell, Dugan, and Stewart's recent book this argument is put forward, and it is estimated that probably the average diameter of the particles in the nucleus of a comet is about 1 cm. at least.§ If the nucleus were made up of solid or liquid particles much smaller than this, or of gases, the influence of radiation-pressure on its orbit would be apparent in the observations.

The relation which appears to exist between meteoric showers and the orbits of certain comets is further evidence of their meteoric constitution.

The present paper deals with the production and intensity of line and band spectra in comets.

* A. Fowler, *M.N.R.A.S.*, **70**, 176, 484 (1910).
† *Loc. cit.*, p. 495.
‡ Excepting the apparent deceleration experienced by Encke's Comet.
§ Russell, Dugan, and Stewart, *Astronomy* (1926), i. p. 443.

Quantitative determinations of the intensity of luminosity, as far as is known, have been confined mainly to the total luminosity of the comet's head and, in some cases, to that of its nucleus. The outstanding work in this field, based on a critical discussion of the available data, was done by Holetschek on comets observed in more than one apparition. In the present treatment, Holetschek's values for the visual brightness and for the diameter of the comet's head will be used as a first test for the proposed mechanism of luminosity of the band and line spectra. Since no suitable intensity measurements as regards the spectra are available, it is hard to say which part of the observed brightness is due to the continuous spectrum and which part is due to the bright band and line spectrum, but in cases where the two spectra are comparable it is safe to conclude that Holetschek's value gives also the proper order of magnitude for the brightness due to either continuous spectrum or line and band spectrum.

The explanation of the continuous spectrum of a comet as due to reflection of sunlight by solid bodies or solid particles does not seem to meet with serious difficulties. In particular the observed brightness of the head is easily accounted for without necessitating a too large mass of the comet.*

The mechanism of luminosity which causes the band and line spectra in the comet's head must be of an entirely different nature. Up to now it has usually been assumed that these spectra are produced by electrical excitation of a nature similar to that causing band and line spectra in a discharge tube.† In the present paper a different mechanism is worked out, namely, that these spectra are produced by ordinary excitation of the atoms and molecules in the comet's head through the sunlight. This mechanism was first proposed by Schwarzschild and Kron ‡ in their work on the tail of Halley's Comet as being responsible for the luminosity in the tails of comets. It is possible to calculate the intensity of the spectra which this mechanism of excitation is able to produce. It will be shown that the luminosity of which the mechanism is theoretically capable is larger than the observed luminosity given by Holetschek, so that the mechanism proposed appears more than sufficient to account for the observed luminosity of the band and line spectra for comets observed in more than one apparition.

There are, however, a number of comets which have been observed in one apparition only, and which by their extreme brightness would appear to furnish a more severe test of the proposed mechanism. Three representatives of these comets are discussed in section 4, and the observational data have been taken directly from the available literature. It will appear that in one case, the Comet Wells, the calculated luminosity due to the sodium lines, as far as can be seen, agrees roughly with the observed luminosity.

* Russell, Dugan, and Stewart, *Astronomy* (1926), i. pp. 430, 443, 444. Compare, however, section 5.

† B. Hasselberg, *Ast. Nach.*, **104**, 14 (1883).

‡ See section 5.

2. *Visibility Data used. Production of a Bright Line Spectrum by Resonance. Expression for the Monochromatic Brightness of the Comet's Head.*

In deriving an expression for the visual brightness of the comet's head, the different sensitivity of the eye for light of different wave-lengths will be taken into account. This can be done in a quantitative manner by making use of visibility data determined in the laboratory, an extended discussion of which data was given by Gibson and Tyndall.* The values of the visibility for light of various wave-lengths which those authors recommend are given in Table I. The first column contains the wave-length λ expressed in microns, the second column the corresponding value of the visibility β_e, defined as visual brightness divided by energy, and the third column the visibility per quantum β, defined as the visual brightness divided by the number of quanta. Since only the ratio of the visibilities for light of different frequencies is to be considered, both β_e and β have been expressed as fractions of their maximum value.

TABLE I.

Visibility Data.

λ.	β_e.	β.	λ.	β_e.	β.
0·40	0·0004	0·0006	0·59	0·757	0·709
·41	·0012	·0016	·60	·631	·582
·42	·0040	·0053	·61	·503	·456
·43	·0116	·0149	·62	·381	·340
·44	·023	·029	·63	·265	·232
·45	·038	·047	·64	·175	·151
·46	·060	·072	·65	·107	·091
·47	·091	·107	·66	·061	·051
·48	·139	·160	·67	·032	·026
·49	·208	·235	·68	·017	·014
·50	·323	·357	·69	·0082	·0066
·51	·503	·545	·70	·0041	·0032
·52	·710	·755	·71	·0021	·0016
·53	·862	·899	·72	·00105	·00081
·54	·954	·976	·73	·00052	·00039
·55	·995	1·000	·74	·00025	·00019
·56	·995	0·982	·75	·00012	·00009
·57	·952	·923	·76	·00006	·00004
·58	·870	·829			

λ represents the wave-length in microns, β_e the visibility in the ordinary sense, β the visibility per quantum.

* K. S. Gibson and E. P. T. Tyndall, *Scientific Papers of the Bureau of Standards*, **19**, 131 (1923), No. 475.

The values of β_e are those recommended by Gibson and Tyndall. Since the energy of one quantum is $h\nu$, the visibility of one quantum is $\beta_e h\nu = \beta$, and, if expressed in an arbitrary unit, $\beta = \text{const. } \beta_e/\lambda$. The values in the third column are therefore obtained from those in the second by dividing by λ and then multiplying by a constant such as to make the maximum β equal to unity. The maxima of β_e and β occur for almost the same wave-lengths, namely, 0·555 μ and 0·550 μ respectively.

The production of a bright line spectrum in the comet's head by sunlight will now be considered. Disregarding, for the time being, the existence of absorption lines in the sun's continuous spectrum, the sun may be regarded as a spherical black body of surface temperature T = 6000° C. Abs. and radius R. The total radiation emitted per second by the sun within the frequency interval $d\nu$ is accordingly *

$$\pi R^2 c\rho_\nu d\nu \quad . \quad . \quad . \quad (1a) \qquad \rho_\nu = \frac{8\pi h}{c^3}\nu^3 e^{-\frac{h\nu}{kT}} \quad . \quad . \quad (1b)$$

$$h = 6{\cdot}56 \times 10^{-27}, \qquad k = 1{\cdot}372 \times 10^{-16}.$$

The quantity $\rho_\nu d\nu$ represents the energy density of black body radiation, and the expression (1*b*), which is in the form of Wien's Law, holds with sufficient accuracy for the visual and shorter wave-lengths, and is obtained in the well-known way by omitting the term unity in the numerator of Planck's Law, which approximation is justified for T = 6000°.

The total number of quanta emitted by the sun per second within $d\nu$ is then

$$\frac{8\pi^2 R^2 k^3}{c^2 h^3} T^3 x^2 e^{-x} dx \quad . \quad . \quad (2) \qquad x = \frac{h\nu}{kT} \quad . \quad . \quad . \quad . \quad (2a)$$

as is obtained through division of the energy (1*a*) by $h\nu$, and substitution of (1*b*).

If this number of quanta (2) is multiplied by the visibility per quantum β, their visual brightness is obtained, and subsequent integration over all frequencies ν from zero to infinity gives the total visual brightness of the sun.

The sun's absolute visual brightness B is therefore

$$B = \frac{8\pi^2 R^2 k^3}{c^2 h^3} T^3 \int_0^\infty \beta x^2 e^{-x} dx \quad . \quad . \quad . \quad (3)$$

The gases and vapours which from its spectrum are known to be present in the comet's head will be capable of absorbing spectral lines from the sunlight which passes through them. If, *e.g.*, sodium vapour is present, the D-line of sodium will be absorbed from the sun's spectrum. In general, let the comet be capable of absorbing a spectral line α of which the centre has a frequency ν_α. The absorption then takes place in a certain narrow region about ν_α, and it is convenient to introduce

* Compare the similar expressions (1*a*) to (2*a*) in *Ap. J.*, **65**, 53 (1927).

for this absorbed spectral line a certain effective width $\Delta\nu_a$, meaning by this that the amount of light absorbed is the same as if the absorption were complete within the region $\Delta\nu_a$ and absent outside this region.* The quantity $w = \Delta\nu_a/\nu_a$ will be designated as the relative effective width, for which holds

$$w = \frac{\Delta\nu_a}{\nu_a} = \frac{\Delta x_a}{x_a} \quad . \quad . \quad . \quad . \quad (4)$$

according to (2*a*).

Let the comet's head subtend a solid angle Ω as seen from the sun's centre. Then the fraction $\Omega/4\pi$ of the sunlight is intercepted by the comet, and of this fraction only an amount corresponding to the frequency interval $\Delta\nu_a$, the effective width, is absorbed. The number of quanta absorbed by the comet's head in the spectral line a is therefore

$$N_a = \frac{\Omega}{4\pi}\frac{8\pi^2R^2k^3}{c^2h^3}T^3x_a{}^2e^{-x_a}\Delta x_a \quad (5), \qquad \Delta x_a = wx_a \quad . \quad . \quad (5a)$$

according to (2) and (4).

If a is a resonance line in the visual region, this same number is re-emitted by the comet's head as a bright line of the same frequency, where, for the sake of simplicity, the re-emission will be considered to occur evenly in all directions, so that the calculated luminosity of the comet is an average value as regards direction. The visual brightness of the comet's head in the light of this resonance line is therefore

$$B_a = \beta_a N_a \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where β_a represents the visibility of one quantum of frequency ν_a.

From (3), (5), (5*a*), and (6) there follows for the ratio of the absolute visual brightness of the comet's head in the light of the resonance line a to the absolute visual brightness of the sun

$$\frac{B_a}{B} = \frac{\Omega}{4\pi}\frac{\beta_a x_a{}^3 e^{-x_a} w}{\int_0^\infty \beta x^2 e^{-x} dx} \quad . \quad . \quad . \quad . \quad (7)$$

Without loss in generality we can assume the comet to be one astronomical unit away from sun and earth.† Let D_1 be the apparent diameter of the comet's head in minutes of arc viewed from this distance. The linear diameter is then $d = 2\pi D_1/(360 \times 60)$ astronomical units. On a sphere of radius 1 astronomical unit surrounding the sun the comet's head covers an area $\pi d^2/4 = \Omega$, from which

$$\frac{\Omega}{4\pi} = \frac{D_1{}^2}{1{\cdot}891 \times 10^8} \quad . \quad . \quad . \quad . \quad (8)$$

* A rather wide spectral line which is incompletely absorbed could therefore have a small effective width.

† This is done, since Holetschek's observational values for brightness and diameter of the head have been reduced in this way according to the inverse square law. Following Holetschek, the suffix 1 has been used to indicate this reduction to distances of one astronomical unit.

In evaluating the integral in the denominator of (7) a graphical method has been used. Values of x from $x = 3{\cdot}2$ ($\lambda = 0{\cdot}745$) to $x = 5{\cdot}8$ ($\lambda = 0{\cdot}411$), with intervals of 0·1, were plotted against the integrand and the area under the curve determined. In evaluating the integrand for each value of x, the corresponding λ was first found by means of the relation $\lambda = hc/(k\mathrm{T}x) = 2{\cdot}383/x$, for $\mathrm{T} = 6000°$, which readily follows from (2*a*), and then the quantity β obtained from Table I. by graphical interpolation. The result of this quadrature is

$$\int_0^\infty \beta x^2 e^{-x} dx = 0{\cdot}205 \quad . \quad . \quad . \quad . \quad (9)$$

the error introduced in replacing the limits zero and infinity by the two values mentioned above being insignificant.

Substituting (8) and (9) in (7), the following formula for the ratio of the monochromatic brightness of the comet's head to the sun's visual brightness is obtained :

$$\frac{\mathrm{B}_\alpha}{\mathrm{B}} = \frac{\beta_\alpha \mathrm{D}_1^2 x_\alpha^3 e^{-x_\alpha} w}{3{\cdot}88 \times 10^7} \quad . \quad . \quad . \quad . \quad (10)$$

This being the ratio of their intrinsic brightnesses, is also the ratio of their brightnesses viewed at the same distance of 1 astronomical unit.

If, at a distance of 1 astronomical unit, the monochromatic brightness of the comet's head in magnitudes be $m_{1\alpha}$, and the sun's brightness expressed in the same way be $m_\odot$, we have the reduction

$$m_{1\alpha} - m_\odot = 2{\cdot}5 \log \frac{\mathrm{B}}{\mathrm{B}_\alpha} \quad . \quad . \quad . \quad (11)$$

Substituting the value $m_\odot = -26{\cdot}7$ for the observed relative magnitude of the sun, equations (10) and (11) lead to

$$m_{1\alpha} = -7{\cdot}7 - 5 \log \mathrm{D}_1 - 2{\cdot}5 \log \beta_\alpha - 2{\cdot}5 \log w + 1{\cdot}086 x_\alpha - 7{\cdot}5 \log x_\alpha \quad (12)$$

which is the desired expression for the monochromatic visual brightness $m_{1\alpha}$ of the comet's head in stellar magnitudes at a distance of 1 astronomical unit. The spectral line α considered is a resonance line.

In certain cases the spectral line produced might not be a resonance line, the mechanism is then not that of resonance but that of fluorescence. Let the absorbed spectral line be γ, the emitted spectral line which is of longer wave-length be α, as before. After a quantum has been absorbed in the line γ, in precisely the same way as has been discussed, the electron drops back to lower levels, and in doing so, on its way to the lowest level in the atom, may emit the spectral line α; hence, for every quantum absorbed only f quanta are re-emitted in this line, where f is smaller than one, but perhaps of the order 1.

The final expression for the monochromatic brightness of the comet's head in the light of line α, due to the absorption in the line γ, becomes then

$$m_{1\alpha} = -7{\cdot}7 - 5 \log \mathrm{D}_1 - 2{\cdot}5 \log f\beta_\alpha - 2{\cdot}5 \log w + 1{\cdot}086 x_\alpha - 7{\cdot}5 \log x_\alpha \quad (12a)$$

which reduces to (12) for the special case when $\alpha = \gamma$ and $f = 1$.

The previous derivations may easily be carried out under the assumption that the comet is r astronomical units from the sun and Δ astronomical units from the earth. If, then, m_α represents the monochromatic apparent brightness of the comet, D its apparent diameter in minutes of arc viewed from the distance Δ, the same formulæ (12) and (12a) may be retained, provided that

$$m_{1\alpha} = m_\alpha - 5 \log r - 5 \log \Delta \quad . \quad . \quad . \quad (13)$$

$$\log D_1 = \log D + \log \Delta, \quad \text{or} \quad D_1 = D\Delta \quad . \quad . \quad (14)$$

The quantity $m_{1\alpha}$ is therefore nothing but the apparent brightness m_α of the comet's head, reduced to $r = 1$, $\Delta = 1$, according to the inverse square law, and D_1 is the apparent diameter D reduced to $\Delta = 1$. In both cases the reduction is indicated by the suffix 1.

This method for reducing the observations of m and D, according to (13) and (14), was used by Holetschek, and hence Holetschek's reduced values may be used for comparing the theoretical expression (12), or (12a), with the observations.

3. *Comparison with Observation: Periodic Comets.*

The widening of spectral lines may be due to different causes, and hence the effective width of a spectral line absorbed by gases in the comet's head may have different values depending on the conditions present. One cause for widening which is always active in a gas of temperature T′ is the Doppler effect due to the thermal motion of the molecules. If only this factor is taken into consideration the kinetic theory shows that in a spectral line α the absorption by a thin layer of gas is a maximum for the centre of the line ν_α, and is reduced to one-half of its maximum value for the frequencies $\nu_\alpha + \frac{1}{2}\Delta\nu_\alpha$ and $\nu_\alpha - \frac{1}{2}\Delta\nu_\alpha$, given by

$$w = \frac{\Delta\nu_\alpha}{\nu_\alpha} = 0{\cdot}718 \times 10^{-6}\sqrt{\frac{T'}{a}} \quad . \quad . \quad . \quad (15)$$

where a is the molecular or atomic weight. The quantity $\Delta\nu_\alpha$ may be termed the width of half intensity and w the relative width of half intensity according to the kinetic theory. If so much gas is present that the absorption is practically complete for the centre ν_α of the line, one may take it to be complete within the width of half intensity $\Delta\nu_\alpha$, but proceeding farther outward from the centre the absorption will soon decrease rather rapidly and become negligible at a distance comparable with $\Delta\nu_\alpha$. In other words, as regards order of magnitude, the effective width of absorption is then $\Delta\nu_\alpha$ and the relative effective width used in (12) or (12a) is w, given by (15). Since the condition of practically complete absorption at the centre of a line does not appear to require an excessive mass of gas in the comet's head, it seems permissible to use (15) in calculating by means of (12) or (12a) the luminosity of which the theoretical mechanism discussed in section 2 is

capable. At the same time it must be realised that, if other factors determining the width of the lines are taken into consideration, the luminosity may become still larger than that calculated from the kinetic theory width. This matter will again be taken up in section 4.

The temperature T′ of the gases in the comet's head presumably corresponds to the temperature of the meteorite from which the gases have been freed, and an approximate value for it may therefore be obtained in the following way: If the sun has a radius R and a temperature T, and the meteorite at a distance r is treated as a sphere of radius R′ and a practically uniform temperature T′, Stephan's Law applied to the sun's radiation gives for the amount of energy intercepted by the meteorite $a\mathrm{T}^4\pi\mathrm{R}'^2\mathrm{R}^2/r^2$, a being the constant of the Stefan Boltzmann Law. Considering the meteorite as a black body, this energy intercepted is completely absorbed, and then re-emitted as black body radiation of temperature T′, the amount of which, according to the same law, is $a\mathrm{T}'^4 4\pi\mathrm{R}'^2$. Equating these two amounts we obtain $\mathrm{T}' = \mathrm{T}\sqrt{\mathrm{R}/2r}$, or, since T = 6000° and R = 0·00465 astronomical units,

$$\mathrm{T}' = \frac{289^\circ}{\sqrt{r}} \quad . \quad . \quad . \quad . \quad (16)$$

where r is the distance of the comet's head from the sun, expressed in astronomical units.

Since (16) gives the temperature of the comet's head and (15) the relative effective width for this temperature, the monochromatic brightness of a comet's head which the mechanism is capable of producing may now be calculated for different spectral lines, according to (12) or (12a), the assumption being made that the absorption in the centre of the spectral line is complete and the only widening taken into consideration being that given by the kinetic theory. In Table II. this has been done for a temperature T′ of 289°, corresponding to a distance of 1 astronomical unit, which is of the order of the perihelion distance for the periodic comets which will presently be discussed. The values given are those for $m_{1a} + 5 \log \mathrm{D}_1$, which quantity may also be considered as the monochromatic brightness of a comet of apparent diameter 1′. In every case the brightness due to one single line is calculated, so that the total intensity of the Swan and CO bands is much larger. For the Swan bands the value 24 for the molecular weight a corresponding to C_2 has been used, and they have been treated as resonance bands. This is presumably justified since they are either that or at any rate require low excitation, as Professor Birge informed me. The Ångstrom bands due to CO were interpreted by Birge.* They are not resonance bands having an excitation potential of 10·73 volts (λ 1150 Å) for the exciting frequency of the O–O band, and formula (12a) has therefore been used. The cyanogen bands have not been included, since they are in a region to which the human eye is little sensitive and thus have not been observed visually in comets.

* R. T. Birge, *Physical Review*, **28**, 1157 (1926).

TABLE II.

Theoretical Values of $m_{1a}+5 \log D_1$.

	Na ($a=23$).	Swan Bands ($a=24$).			Ångstrom CO ($a=28$).
		Blue.	Green.	Yellow.	
λ^a	5893	4737	5165	5635	5610
β_a	0·720	0·124	0·700	0·965	0·982
λ_γ	..	..	..	..	1150
$m_{1a}+5 \log D_1$	$6{\cdot}5^m$	$8{\cdot}7^m$	$6{\cdot}6^m$	$6{\cdot}1^m$	$19{\cdot}0^m$

For a given line, β_a is found from Table I., and x_a or x_γ from $x = h\nu/kT = 2{\cdot}383/\lambda$.

TABLE III.

Observed Values of $m_1+5 \log D_1$.

Comet.	U.	q.	D_1.	m_1.	$m_1+5 \log D_1$.	Spectrum.
			′		m	
Halley	76·5	0·59	3·9	3·5	6·4	(1)
Olbers	72·5	1·20	3·1	4·7	7·1	(2)
Pons-Brooks	71·4	0·78	3·5	4·4	7·1	(3)
Faye	7·5	1·65	1·7	5·5	6·6	(4)
Brooks	7·1	1·96	3	8·4	10·6	(5)
Encke	3·3	0·34	2·2	6·3	8·0	(6)

U represents the period in years, q the perihelion distance in astronomical units, D_1 the apparent diameter in minutes viewed at a distance of 1 astronomical unit, and m_1 the brightness of the head in magnitudes reduced to $r = 1$ and $\Delta = 1$.

(1) April–May 1910, V. M. Slipher, *Lowell Obs. Bulletin*, No. 52, vol. xi., No. 2 (1911).
(2) Last apparition 1887. No observations of spectrum available.
(3) September 1883, V. Konkoly, *Ast. Nach.*, **107**, 41 (1884).
(4) Only in apparition of 1843 the tail was well developed.
(5) May–June 1890, A. Fowler, *Nature*, **42**, 112, 162 (1890).
(6) 1886, Bredichin-Jaegermann, *Mechanische Untersuchungen über Cometenformen* (1903), p. 50.

Table III. gives the observational material used for testing the theory which is taken from Holetschek's work on the periodic comets observed in more than one apparition.* The brightness m (in Holetschek's notation H) is that obtained by comparing the comet with stars of known magnitude in the field of the smallest telescope available or with the naked eye,† and the reduced value m_1 (Holetschek's H_1) refers, therefore, to the integrated light of the comet or of its head, the difference being immaterial for our purpose. The observed apparent diameter D, which after reduction becomes D_1, refers also to the head

* J. Holetschek, *Denkschriften der Wiener Akademie*, **93**, 201 (1917); **94**, 375 (1918), and references contained there.
† *Denkschriften*, **88**, 854 (1913).

and not to the nucleus.* Its value is rather inaccurate, and even somewhat arbitrary, since the light of the head gradually fades out and its boundary is therefore not sharp. The formulæ for reduction used by Holetschek are $m_1 = m - 5 \log r - 5 \log \Delta$, and $D_1 = D\Delta$, corresponding to (13) and (14) of section 2; but it should be kept in mind that the observed m_1 refers to the total luminosity of the head, whereas m_{1a} in (13) is the monochromatic theoretical brightness. For the complete material the reader is referred to the original work, in particular to the table on p. 488 in *Denkschriften*, **94**, which gives a complete summary. For the present discussion only those comets have been selected having a tail larger than 0·01 astronomical unit, since they are the more luminous and the line and band spectra are usually best developed. Among these, two comets have been discarded, Westphal's Comet and Holmes' Comet, the former since in the one apparition with large tail development (1913) it diminished in brightness and had disappeared completely before it reached perihelion, the latter, since it had a large tail developed only during two abnormal increases in luminosity in 1892, and its spectrum was then continuous. The values for $m_1 + 5 \log D_1$ have been given near or at perihelion, and represent the maximum luminosity. With comets which became fainter in successive apparitions care has been taken to use the value for the luminosity in the first apparition, and, if possible, the value of the apparent diameter has been taken to correspond to this time of maximum luminosity. In this way the present values may differ slightly, though not materially, from the values given in Holetschek's summary.

The brightest comet in Table III. is Halley's Comet, having an observed value of $6{\cdot}4^m$ for $m_1 + 5 \log D_1$. The Swan bands are the most prominent feature in the spectrum of a periodic comet, apart from the continuous spectrum,† and it will be noted from Table II. that the absorption of one line in the green or the yellow Swan band would give for $m_{1a} + 5 \log D_1$ a theoretical value of $6{\cdot}6^m$ and $6{\cdot}1^m$, and hence be practically sufficient to account for the total brightness of the head. Since each band is produced by the absorption of a great number of lines, and moreover part of the observed luminosity is due to continuous spectrum, it must be concluded that the absorption of the lines in the comet's head is very incomplete, presumably on account of the small absorption coefficient of the lines in the bands and the moderate concentration of the gas freed. A complete absorption would make the comet's head shine with a brightness much stronger than is observed for the periodic comets.

Because of the presence of CO^+ ions in the tail it might be expected that there is neutral carbon-monoxide in the head. Fowler called attention to the fact that, notwithstanding this, the Ångstrom and Thalen CO bands, which are always obtained in a discharge-tube with

* *Denkschriften*, **63**, 333 (1898).

† The reader is referred to the literature in the footnotes to Table III. An extended treatment of cometary spectra, with bibliography, is given by F. Baldet, *Annales de l'Observatoire d'Astronomie Physique de Paris*, Tome vii., p. 1 (1926).

carbon-monoxide, have not been very definitely recorded in comets. Since from Birge's work these bands are known to have an exciting wave-length of 1150 Ångstrom for the 0–0 band, it is possible to calculate their theoretical intensity in the comet's head. This has been done by means of (12*a*), and the result is given in Table II. for the intensity due to the absorption of one line for the band with the head at 5610 Å, which would be the brightest according to (12*a*). The values $f = 1$ and $\lambda_\gamma = 1150$ Å have been used so that the calculated value represents an upper limit to the brightness. The result given in Table II. means that $m_{1\alpha} + 5 \log D_1$ for one absorbed line is fainter than $19{\cdot}0^m$, or, all circumstances being the same, one might expect the brightest Ångstrom band to be about 13 magnitudes fainter than the brightest Swan band, or about one hundred thousand times as faint. This gives a satisfactory explanation of the fact that the Ångstrom bands have never been definitely observed. If the mechanism of excitation were of an electrical nature, this fact would be hard to understand, and the only means of explaining it would seem to be the assumption that the exciting electron would have enough kinetic energy to produce the Swan bands, but not enough to excite the Ångstrom bands, for which assumption there seems no adequate justification.

The cyanogen bands in the head are very prominent photographically, but not visually. They have therefore not been discussed, but the fact that they are resonance bands fits in quite well with the present explanation. Though no quantitative treatment can be given, the statement can confidently be made that on the basis of the present theory all bands prominent visually or photographically must be either resonance bands or bands of low excitation, which is also in agreement with the data regarding the spectrum of the tail.*

The band 4300 due to hydrocarbon is very likely a resonance band, or at any rate a low excitation band,† and its occurrence in comets would therefore be expected if hydrocarbons are present. Baldet,‡ however, rejects the identification of the image 4314, present, *e.g.*, in the nucleus of Brooks' comet, with this band, since the observed image does not degrade towards the violet.

4. *The Comet Wells of* 1882 *and some other Comets observed in One Apparition.*

In section 3 the conclusion was reached that for the periodic comets the absorption in the lines and bands is incomplete on the basis of the proposed mechanism, which therefore is amply sufficient to account for the observed brightness.

The hope for a more severe test of the theory seems to lie in those comets observed in one apparition, which came very near to the sun and then were exceedingly bright. Since it is much more likely that the absorption becomes complete in the line spectrum of an element

* See section 5.

† R. S. Mulliken, *Physical Review*, **30**, 808 (1927).

‡ F. Baldet, *loc. cit.*, p. 41.

than in a band spectrum, on account of the very small absorption coefficient of the latter, it seems best to concentrate on the line spectrum of sodium, and perhaps that of iron, produced in some of these comets.

Not much of the material available is fit, even for a roughly quantitative test, since simultaneous observations regarding spectrum, diameter of head, and brightness are rare, and in many cases it is hard to decide how much of the luminosity is due to the line spectrum. A crucial case, however, seems to be presented by the Comet Wells of 1882, which is the first comet in which the D-lines of sodium were observed. The observations to be discussed are those made on 1882 June 6 by Copeland and Lohse at Dun Echt * and by Vogel in Potsdam.†

In the first place, it may be concluded that at that date the visual light of the comet was to a great extent due to the D-lines of sodium. "With a wide slit," Copeland and Lohse remark, "the sight was really magnificent: exactly like a prominence, the perfect image of the comet, head and tail, being visible in the light of the D-line, very bright and clear." Their drawing of the slitless spectrum in Plate IV., fig. 6, shows very little continuous spectrum.

Vogel remarks that on June 5 the nucleus and adjoining envelope appeared strongly yellow, and further: "On June 6 the intensity of the yellow line in the cometary spectrum was so great that one succeeded, with a spectral apparatus of high dispersion, in recognising it as clearly double and coinciding with the artificial sodium line." "The light emerging from the incandescent sodium vapours exceeded to such an extent the rest of the comet's own and reflected light, that the comet appeared yellow without the spectroscope. . . ."

In the second place the observations show that the two sodium lines were very wide and diffuse at the edges. The abnormally large width must be considered as well established, since the lines were compared with the artificial sodium lines. The effect is not confined to June 6, but seems to have persisted throughout the observations. It was so evident that Copeland and Lohse seemed at first to hesitate in attributing the cometary lines to sodium, and they state explicitly that "none of the suspected lines ever attained the sharpness of D. . . ." On June 6, "just at sunrise, with a very narrow slit, the D-line towards the violet is fully twice as long as the other; both lines appear much thicker in the nucleus." From their drawing of this spectrum one may conclude that the width of the two D-lines together in the densest part is comparable with the distance between them, which is 6·0 Å. This leads to a relative effective width w of order 1/1000 to be substituted in the expression (12). In some of their other drawings of cometary spectra the D-lines are quite well defined, so that the procedure followed here seems justified.

Vogel's evidence on June 6 is of the same nature. "The two sodium lines of the cometary spectrum were of very unequal intensity, the more refrangible line appeared strongly widened, about five times as wide, diffuse at the edges and much stronger than the less refrangible."

* Ralph Copeland and J. C. Lohse, *Copernicus*, **2**, 225 (1882).
† H. C. Vogel, *Ast. Nach.*, **102**, 199 (1882).

The diameter of the head of the comet on June 6, or rather of that part for which the sodium emission is strongest, may be estimated from Copeland and Lohse's drawing of the slitless spectrum in Plate IV. of their paper. Though this is not explicitly mentioned, the Grubb spectroscope (long collimator, Grubb compound prism, minimum deviation for F) was no doubt on the 15-inch refractor, and in this case the length of the slit is $1\frac{1}{2}$ minutes of arc (p. 235), which corresponds to the height of the drawn spectrum or 1 cm. The bright knot, which is the monochromatic picture of the head, has a height of 1·3 mm. in the drawing, or 0·2′. Allowing for the horizontal widening due to the double D-line, the horizontal dimension of the knot in the picture, or Copeland and Lohse's statement that this dimension is three to four times the interval between the D-lines, gives a horizontal extension of the head, which is roughly the same, and hence a value of 0·2′ will be adopted for D.

The brightness of the head was estimated by Copeland and Lohse simultaneously with the observations of the spectrum just referred to. "Dr. Copeland estimated the brightness of the comet equal to 1·5th magnitude, and I (Lohse) found for it quite independently as limits, between which it must be classified, the first and second magnitude." Hence $m = 1 - 2$. Since part of the brightness may be due to continuous spectrum, and in view of the inaccuracy of the estimates, $m = 2$ will be taken as the brightness to be accounted for.

For r and Δ the values of Lamp * may be used, retaining only two decimal places. At perihelion (June 10) $\log \Delta = 0{\cdot}02$, $\log r = 8{\cdot}78$, $T' = 1180°$ Abs. On June 6: $\log \Delta = 0{\cdot}00$, $\log r = 9{\cdot}41$, $T' = 570°$ Abs., the values of T' being readily obtained from (16). After reduction of $m = 2$ and $D = 0{\cdot}2'$ to $r = 1$ and $\Delta = 1$, by means of (13) and (14), the observational data for the Comet Wells on June 6 may be summarised as follows:

$$m_1 = 4{\cdot}4, \quad D_1 = 0{\cdot}2', \quad m_1 + 5 \log D_1 = 0{\cdot}9, \quad w \text{ of order } 1/1000.$$

The light of the comet being to a great extent monochromatic, one may put $m_1 = m_{1a}$, and hence $m_{1a} + 5 \log D_1 = 0{\cdot}9$. Substituting this value in (12), with the values of x_a and β_a for the sodium resonance line, and solving for w, one obtains $w = 0{\cdot}0004$, which is of the order 1/1000. Hence one may conclude that, in order to account for the observed values of luminosity and diameter, the theory requires the rather large relative effective width $w = 0{\cdot}0004$ or an effective width $\Delta\lambda = 2{\cdot}4$ Å, which is of the same order as indicated by Copeland and Lohse's and Vogel's observations. In view of the inaccuracy of the estimates, the check of the theory furnished by the observations may be considered satisfactory.

It remains to find a physical reason for the abnormally large effective width which does not lead to a too large mass for the sodium vapour in the comet's head. For $T' = 570°$, the kinetic theory formula (15), with $a = 23$, leads to $w = 3{\cdot}6 \times 10^{-6}$, hence the Doppler effect of thermal motion does not give the required width. In recent years,

* E. Lamp, *Ast. Nach.*, **102**, 35.

however, the theory of widening of spectral lines due to radiation damping has met with success in cases where lines are abnormally wide. Minkowski * has observed this effect quantitatively for the absorbed D-lines of sodium in the laboratory, whereas astrophysical applications to absorption lines were made by Stewart,† by von Klüber,‡ and by Unsöld.§

It may be recalled that in classical electrodynamics the equation of motion of an elastically isotropically bound electron (or space oscillator) contains a damping term due to the radiation emitted by the electron. Let the frequency without damping term and without incident radiation be ν_a, the electron's own frequency. If there is a beam of incident radiation of frequency ν in the neighbourhood of ν_a, and cross-section 1 cm^2, the damping term in the equation of motion causes the incident radiation to be scattered, the fraction scattered being

$$\sigma dx = \frac{2\pi\epsilon^4}{3m^2c^4}\frac{\nu_a^2}{(\nu-\nu_a)^2}\mathrm{N}dx \quad . \quad . \quad . \quad (17)$$

where dx is the distance travelled by the parallel incident beam, N the number of oscillators per cm^3., ϵ the electronic charge, m the electronic mass, and c the velocity of light.‖ Let N be constant, then the fraction of the energy scattered, and therefore re-emitted in the bright line, is $1 - e^{-\sigma l}$, where l is the length of the layer. Since σ is the function of $\nu - \nu_a$ expressed by (17), the energy scattered is a maximum for the centre of the line, $\nu = \nu_a$, and decreases symmetrically away from the centre. As regards order of magnitude, one may take as the effective width of absorption $\Delta\nu$ that portion of the line contained between the two values of ν for which the energy scattered is half of the incident energy, or $\sigma l = \log_e 2$, hence, from (17),

$$\mathrm{N} = \frac{3m^2c^4}{2\pi\epsilon^4}(\tfrac{1}{2}w)^2 \cdot \frac{1}{l}\log_e 2, \qquad w = \frac{\Delta\nu}{\nu_a} \quad . \quad . \quad (18)$$

remembering that $\nu - \nu_a = \pm\frac{1}{2}\Delta\nu$. Formula (18) gives the order of the concentration N required to produce a relative effective width w, if the beam of light passes through a length l. In the present case $w = 0{\cdot}0004$. The comet had an apparent diameter $0{\cdot}2'$ viewed at a distance of 1 astronomical unit; this gives a linear diameter of $8{\cdot}7 \times 10^8$ cm., which may be substituted for l. Since further

* R. Minkowski, *Zeit. f. Phys.*, **36**, 839 (1926).

† J. Q. Stewart, *A.J.*, **59**, 30 (1924).

‡ H. von Klüber, *Zeit. f. Phys.*, **44**, 481 (1927).

§ A. Unsöld, *Zeit. f. Phys.*, **44**, 793 (1927). Milne, *M.N.*, **88**, 202, agrees with Unsöld's explanation of the widening as due to radiation damping, but offers criticism of his treatment of the solar chromosphere.

‖ Compare formula (1) in Unsöld's paper, *loc. cit.* The formula applies only for that part of the line sufficiently far outside the kinetic theory width (15) and the natural width of half intensity for an infinitesimal layer, which justifies the treatment that is to follow. It is most easily obtained by considering the motion of an electron oscillator under the assumptions stated in the text, the incident radiation being incoherent.

$m = 9{\cdot}05 \times 10^{-28}$, $e = 4{\cdot}775 \times 10^{-10}$, $c = 2{\cdot}998 \times 10^{10}$, $\log_e 2 = 0{\cdot}6931$, one obtains for the concentration of normal sodium atoms in the comet's head

$$N = 2{\cdot}0 \times 10^{8} \text{ atoms per cm}^3\text{., or } 7 \times 10^{-13} \text{ grams per cm}^3. \quad . \quad (19)$$

in which each sodium atom has been treated as a classical oscillator emitting one frequency. The latter approximation is justified as regards order of magnitude.

The values for the comet's diameter and the concentration of sodium just given lead to a mass of normal sodium atoms in the comet's head of the order 4×10^{-14} times the mass of the earth. The total mass of sodium vapour, however, is larger, since the ultra-violet sunlight intercepted by the comet causes ionisation of the sodium atoms. Following considerations by Woltjer,* or by Eddington,† the number N_i of ionised sodium atoms may be calculated as follows, remembering that owing to the high second ionisation potential of sodium only the first ionisation needs to be considered. At the sun's temperature of 6000° the formula used by Saha gives for the equilibrium constant of sodium $N_i^2/N = 2{\cdot}82 \times 10^{16}$, the radiation field ρ_ν is then that of black body radiation (1*b*) for T = 6000°.‡ The radiation field ρ_ν', to which the comet is subjected is, however, this black body radiation, diluted δ times, or $\rho_\nu' = \rho_\nu/\delta$, where $1/\delta = \frac{1}{4}R^2/r^2$, as may be derived from (1*a*). This results in the equilibrium constant being multiplied also by $1/\delta$, or $N_i^2/N = 2{\cdot}82 \times 10^{16}/\delta$. On June 6, $r = 0{\cdot}26$ astronomical unit gives $1/\delta = 3{\cdot}2 \times 10^{-4}$, hence $N_i^2 = 9{\cdot}0 \times 10^{12}$ N. Since, according to (19), $N = 2{\cdot}0 \times 10^{8}$, $N_i = 4{\cdot}2 \times 10^{10}$ ionised atoms per cm³., or, for every normal atom there are about 200 ionised atoms, and certainly not more, since the formula just used involves the assumption that all the recombinations take place on the lowest energy level, thus disregarding capture by higher levels and, moreover, for deeper layers of the comet the applied radiation field may be less than ρ_ν'. The total mass of sodium vapour required is therefore about 200 times that of the mass of the normal atoms just calculated, or 10^{-11} times the mass of the earth. If from gravity considerations a maximum total mass of the comet of perhaps one millionth of the earth is permissible,§ this amount of sodium vapour present at one time in the head is quite conceivable.

It appears, therefore, that the widening due to radiation damping or "natural width" provides an adequate explanation for the large width in the present instance. In addition to this, the fuzziness of the lines observed is also in favour of the explanation offered, since the drop of intensity according to (17) is gradual, being indirectly proportional to the square of the frequency difference. The fuzzy emission lines

* J. Woltjer, jun., *Physica*, **5**, 406 (1925).

† A. S. Eddington, *The Internal Constitution of the Stars*, p. 382 (1926).

‡ It may be remarked that ϱ_ν represents the density per unit frequency of radiation inside a uniformly heated enclosure of temperature 6000°. The actual density at the surface of a spherical black body of surface temperature 6000° is only $\frac{1}{2}\varrho_\nu$.

§ Russell, Dugan, and Stewart, *Astronomy*, i, pp. 429, 443 (1926).

would appear to offer a similar phenomenon to the winged lines in absorption to which Unsöld refers.

Of the other non-periodic comets only two will be briefly considered. The *Great Comet of* 1882 certainly was one of the most conspicuous and is of interest, since on one occasion a number of emission lines attributed to iron were observed by Copeland and Lohse.* Only their observations of the spectrum on September 18 in the daylight will be briefly discussed. For the time of this observation Oppenheim † calculates $\log \Delta = 0{\cdot}0286$, $\log r = 8{\cdot}9286$, from which, retaining only two decimal places, one obtains $r = 0{\cdot}085$ astronomical unit and $T' = 990°$ Abs., according to (16). The continuous spectrum of the nucleus seems to have been rather strong. The most prominent lines in the head were the sodium D-lines, measured as 5889 and 5893 Å, clearly separated.‡ Next in prominence were six lines measured as 5269, 5329, 5369, 5395, 5428, and 5474 Å, which, according to Copeland and Lohse, would correspond to the most prominent iron lines in that part of the spectrum. In Russell's list of ultimate and penultimate lines § the strongest lines of iron (intensity 50 and 60) within the region 5000 to 6000 Å, important for visual observation, are 5269·54 (60), 5328·04 (50), 5371·50 (50), of excitation potential 0·94 volt, and 5615·66 (50) of excitation potential 3·38 volts, for the lowest level. In addition to this, the next lower intensity of 40 includes the lines 5397·14 and 5429·70 of excitation potential 0·94 volt. Hence the five lines observed in the comet are accounted for as being iron lines with their lowest level at 0·94 volt, whereas the sixth line 5474 does not occur in Russell's list, and hence must not be identified as due to iron.

Unfortunately, no reliable indications are available regarding the width of the iron and sodium lines and regarding the observed luminosity due to those lines, and hence the best thing that can be done to check the present theory is to calculate the monochromatic intensity of the comet's head due to each of these lines, assuming the kinetic theory width at a temperature T′ of 990°. Though the result will be of little value regarding the absolute intensities, it should give an idea of the intensities of the iron lines as compared with the sodium lines.

TABLE IV.

Great Comet of 1882 Sept. 18, $T'=990°$ *Abs.* *Kinetic Theory Width* (15) *assumed.*

	Na ($a=23$).	Fe 0·94 V ($a=55{\cdot}8$).	Fe 3·38 V.
λ_α	5893	5269	5616
β_α	0·720	0·870	0·973
λ_γ	..	3760	2210
$m_{1\alpha}+5 \log D_1$	$5^m{\cdot}9$	$6^m{\cdot}9$	$10^m{\cdot}1$

* R. Copeland and J. G. Lohse, *loc. cit.*, p. 234.

† H. Oppenheim, *Ast. Nach.*, **103**, 175.

‡ The values given are averages of the original measurements, which are accurate within a few Ångstroms.

§ H. N. Russell, *Ap. J.*, **61**, 223 (1925).

The intensity calculations were carried out by means of (12) for the sodium resonance lines and by means of (12*a*) for the iron lines with $f = 1$, which seems justified as regards order of magnitude. Table IV. shows that in this way the iron line 5269 is comparable in intensity with each sodium line, though fainter, and the same thing may be said for the other lines with lowest level at 0·94 volt, 5328, 5371, 5397, and 5930, which give also values of about 7·0 magnitudes for $m_{1a} + 5 \log D_1$. The line 5616, which may also be prominent in the visual region, comes out about three magnitudes fainter on account of its higher excitation potential, and hence the fact that it was not observed in the comet may be understood.

Though the observed iron lines were not resonance lines, the foregoing shows that the mechanism of excitation by sunlight may well account for them. It would, of course, be possible that for a sufficiently high temperature T′ many atoms would be in excited states corresponding to 0·94 volt, and thus the lines just discussed might actually become resonance lines. A simple calculation based on statistical mechanics shows, however, that in the present instance where this temperature is 990° Abs., only about one in every 10^5 atoms would be in that state, as compared with the normal state, and hence there can be no doubt that the light produced by resonance from the excited state is negligible as compared with the amount produced by fluorescence, and that expression (12*a*) should be applied. It may also be of interest to remark that Copeland and Lohse observed several bright lines, the identification of which is yet uncertain.

Another very bright comet seen in one apparition was the Comet 1910*a*.* When near the sun it became brighter than Venus. Its spectrum, however, appears to have been mainly continuous, and therefore it seems likely that the part of the luminosity due to line and band spectra was small enough to be produced on the basis of the present theory.

5. *Relation to Work of Others.*

It was shown in 1926 by the author,† that the mechanism of resonance may satisfactorily account for the brightness of the head of periodic comets, as far as it is due to line and band spectra, and the present paper is an extension of this work, including also a case of fluorescence. Schwarzschild and Kron, in 1911, had already proposed the same mechanism for the luminosity of the tail, and their work will presently be referred to.

As regards the continuous spectrum, it may be remarked that the question as to its origin has perhaps not yet quite been settled, though there is much evidence of its being produced by reflection. A difficulty in the reflection theory is, that some very bright comets which showed

* Wright, *L.O.B.*, **5**, 179.

† *Physical Review*, **28**, 428 (1926), Abstract. The formula given there represents only a rough approximation, since for simplicity it was assumed that the visibility per quantum was constant throughout the region 0·39 to 0·76 μ and zero outside this region.

a predominantly continuous spectrum had their brightness increased much faster than would follow from the inverse square law, as they approached the sun. Bobrovnikoff,* from photometric evidence, concludes that, besides a spectrum of the "solar type," the comet's continuous spectrum is in part of a "violet type," with maximum at about 4000 Å, the latter becoming more predominant at distances larger than about 0·7 astronomical unit, but it seems hard to reconcile this result with any explanation from known physical laws.

Schwarzschild and Kron† concluded from their own measurements of intensity along the tail of Halley's comet, combined with Wright's velocity determinations, that the particles which in their motion create the tail have about the same intrinsic brightness, the distance from the sun not varying greatly. In order to explain this result, the authors put forward the theory that the luminosity of the tail is produced by fluorescent or resonance radiation, excited by solar radiation. In particular, they considered the sodium light observed by Wright in the tail of Comet 1910*a* as very probably due to resonance, and assumed an analogous origin for the Fowler CO^+ bands and the Swan bands in the tail of Halley's comet. They also made absolute measurements, and calculated from them the order of the mass lost per second by the comet and the maximum density in the tail.

Though quantitative data regarding the intensity of luminosity in comets' tails are rather scanty, the connected phenomenon of motion of matter has been studied extensively. The observational work by Bredichin and others has established that there is a repulsive force besides that of gravity acting, and this force is now generally assumed to be that of radiation-pressure.‡ Baade and Pauli § have recently calculated the radiation-pressure exerted by sunlight on individual molecules, and found agreement, as regards order of magnitude, with the observational data. It appears, therefore, that the mechanism of absorption and re-emission of sunlight by individual molecules gives not only a satisfactory insight into the luminosity of the head and the tail of comets, but also into the motion of gases in comets' tails.

On the basis of Schwarzschild and Kron's and the present work it may be expected that the prominent bands in the comet's tail are all either resonance bands or bands of low excitation, and this is now known to be the case, since Fowler's CO bands, || due to CO^+, and the negative nitrogen bands,¶ due to N_2^+, are resonance bands.

When the work on the present problem was started at the California Institute of Technology, discussions with Dr. R. M. Badger regarding

* N. J. Bobrovnikoff, *Ap. J.*, **66**, 145, 439 (1927).

† K. Schwarzschild and E. Kron, *Ap. J.*, **34**, 342 (1911).

‡ N. J. Bobrovnikoff, *P.A.S.P.*, **40**, 164 (1928), gives a summary of work done and literature references.

§ W. Baade and W. Pauli, jun., *Die Naturwissenschaften*, **15**, 49 (1927). See also Unsöld's remark (*Zeitschrift für Physik*, **44**, 803, 1927) that the radiation-pressure as calculated by them should be divided by 2, which, however, does not change the order of magnitude.

|| R. T. Birge, *loc. cit.*

¶ R. T. Birge and H. Sponer, *Physical Review*, **28**, 259 (1926).

band spectra were very material in clarifying the author's ideas. Professor R. T. Birge gave very active help in aspects of the problem connected with the band spectra in comets, which was the more appreciated, since some of the material he supplied was then in state of publication. I am greatly indebted to Professor H. H. Plaskett, at present at Harvard, who induced me to introduce the visibility of radiation and who discussed with me the identification of the iron lines.

To these persons in particular I wish to express my sincere thanks.

Summary.

In line with Schwarzschild and Kron's explanation of the luminosity of comets' tails, the theory is proposed that the bright line and band spectra observed in the head of a comet are produced by absorption of sunlight by the gases and vapours in the comet's head and subsequent re-emission of the same wave-length (resonance) or a longer wave-length (fluorescence) and formulæ for the brightness of the comet's head due to one line are derived for both cases.

It is shown that for the periodic comets treated by Holetschek the proposed mechanism is amply sufficient to account for the observed luminosity if the effective kinetic-theory width is assumed for the absorbed spectral line.

In the case of the Comet Wells of 1882 the observed brightness interpreted on the basis of the proposed mechanism requires a width of the sodium lines in that comet comparable with their separation. This is supported by the observed fuzziness of the lines. In parallel with investigations by different authors regarding absorption lines, the very large width of those emission lines is attributed to radiation damping or "natural width," and it is shown that this requires a mass of sodium vapour, normal and ionised, of the order 10^{-11} times that of the earth, which appears permissible.

The iron lines in the Great Comet of 1882, observed by Copeland and Lohse, are not resonance lines; but the mechanism of fluorescence appears to be sufficient to produce the required intensity, since the lines are all produced under low excitation.

The conclusion may be reached that the proposed theory is capable of producing the observed intensity of the line and band spectra in cases where a check has been possible.

The Distribution of Energy near the Limb of the Sun.

By M. Minnaert.

(*Communicated by Prof. W. de Sitter.*)

For many years it has been the general opinion that the brightness of the solar disk decreases gradually towards the limb, and reaches there a value of 0·20–0·50 of the central intensity, according to the wave-length considered; at the extreme limb, the radiation has been

supposed to fall off "suddenly," that is to say, over a distance of $1'' = 0{\cdot}1$ per cent. r it would be reduced in a proportion 100 to 1. This sharp solar limb appeared difficult to explain, and several writers have attempted to give reasons for it. Recently, however, new measurements point to another possibility. The measures of Moll, Burger, and van der Bilt,* suggest that the brightness increases nearly linearly from a value 0 at the extreme limb, at least over 1 per cent. of the radius, probably even further. The slope of the curve $m = \dfrac{di/i_{\text{centre}}}{dr/r}$ seems to be of the order 25–40 according to the wave-length, perhaps 30 for white light. The same result is expressed with positiveness in a paper by Kienle and Juska on this subject.† Their measurements give a slope of 20, while the "effective" wave-length was about 4850 A.U.

We will attempt to show that the old description of the facts was the right one. The curves which show a gradual increase from a value 0 cannot be trusted near the utmost limb; we feel quite sure that the excellent observers quoted have surmounted the experimental difficulties of the subject, but it remains impossible to avoid the scintillation, which must blot out the sharp limb.

Our opinion seems justified by the visual aspect of the sun viewed through a telescope, and by the old results of Vogel,‡ who measured visually a brightness 0·13–0·30 quite close to the limb. It is possible that the eye involuntarily corrects the boiling of the limb, and gives a more exact impression of the object than an automatic instrument can do.

But a real decision can only be obtained by the eclipse method, devised by Julius,§ which is independent of the scintillation. During a total solar eclipse the integrated radiation from the entire uneclipsed part of the disk is measured; the "eclipse-curve" thus obtained gives information about the distribution of the brightness over the disk. A discussion of this method has already shown‖ that the variability of the sky may restrict its utility, but that it retains its value for the very last minutes before totality and the first minutes after it. For this interval the method is excellent, and it seems that the existing data give already a criterion in favour of the existence of a sharp limb.

1. *Formulæ for the Eclipse-curve near Totality.*

We will derive some approximate but very simple formulæ for the eclipse-curve to be expected near totality (fig. 1). The centres of sun and moon are called S and M; the radii, r and $r + \rho$; the distance of the centres, c = SM. The numbers c and ρ are small compared with r.

* *B.A.N.*, No. **91**, 83, 1925.
† *Zs. f. Phys.*, **47**, 426, 1928.
‡ *Ber. Preuss. Akad. Wiss.*, Berlin, 104, 1877.
§ *Ap. J.*, **37**, 225, 1913.
‖ W. H. J. Moll and J. van der Bilt, *B.A.N.*, No. **30**, 170, 1922; W. H. Julius, *ibid.*, No. **33**, 189, 1923.

At a point P, for which the angle PSO $= \phi$, the width of the uneclipsed crescent is

$$\text{PB} = \text{PM} - \text{BM} = \sqrt{r^2 + c^2 + 2rc\cos\phi} - r - \rho \simeq c\cos\phi - \rho.$$

The extremities of the crescent correspond with an angle $\phi^* = \arccos\frac{\rho}{c}$.

In order to calculate the total intensity of the radiation we consider two hypotheses.

Hypothesis A.—The brightness is constant over the marginal portion of the disk and equal to a per unit surface. The radiation of a strip

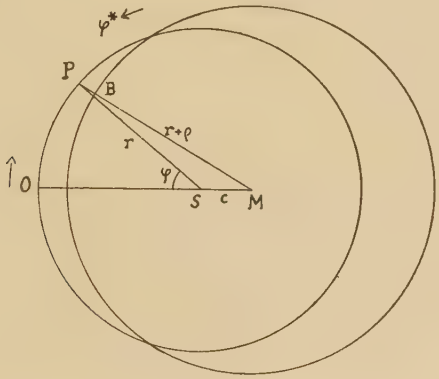


Fig. 1.

from the crescent, length PB and width 1, is $a(c\cos\phi - \rho)$. The radiation of the whole crescent is

$$J_1 = 2\int_0^{\phi^*} a(c\cos\phi - \rho)r\,d\phi = 2a\rho r\left[\sqrt{\frac{c^2}{\rho^2} - 1} - \arccos\frac{\rho}{c}\right].$$

If I is the radiation of the whole disk,

u the proportion $\dfrac{\text{brightness limb}}{\text{brightness centre}}$, v the proportion $\dfrac{\text{brightness centre}}{\text{mean brightness}}$,

we have

(1) . . . . $I = \dfrac{\pi r^2 a}{uv}$,

and

(2) . $\dfrac{J_1}{I} = \dfrac{2uv}{\pi}\dfrac{\rho}{r}\left[\sqrt{\dfrac{c^2}{\rho^2} - 1} - \arccos\dfrac{\rho}{c}\right] = \dfrac{2uv}{\pi}\dfrac{\rho}{r}F_1\left(\dfrac{c}{\rho}\right)$.

The values of v are known with sufficient accuracy; * according to the wave-length they vary from 1·08 to 1·49.

Hypothesis B.—The brightness is proportional to the distance z *from*

* M. Minnaert, *B.A.N.*, No. **51**, 75, 1924; Table IV.

the limb, and equal to bz, this being, however, only valid for the marginal region of the disk.

The radiation of an elementary strip PB is now $\frac{b}{2}\overline{\mathrm{PB}}^2$.

And the total radiation of the crescent :

$$J_2 = b\int_0^{\phi^*}(c\cos\phi-\rho)^2 r d\phi = brc^2\left[\left(\frac{\rho^2}{c^2}+\frac{1}{2}\right)\arccos\frac{\rho}{c}-\frac{3}{2}\frac{\rho}{c}\sqrt{1-\frac{\rho^2}{c^2}}\right].$$

The value of b is expressed as follows in terms of the radiation of the whole disk :—

$$(3)\quad \frac{i}{i_{\text{centre}}}=\frac{mz}{r}=\frac{bz\pi r^2}{v\mathrm{I}};\qquad \text{thus}\qquad b=\frac{m}{\pi r^3}\mathrm{I}v,$$

and

$$(4)\quad \frac{J_2}{\mathrm{I}}=\frac{mv}{\pi}\frac{\rho^2}{r^2}\left[\left(1+\frac{1}{2}\frac{c^2}{\rho^2}\right)\arccos\frac{\rho}{c}-\frac{3}{2}\sqrt{\frac{c^2}{\rho^2}-1}\right]=\frac{mv}{\pi}\frac{\rho^2}{r^2}\mathrm{F}_2\left(\frac{c}{\rho}\right).$$

2. *Numerical Calculations.*

The important fact thus is seen, that in both hypotheses *the decrease of the radiation during the last minutes before totality is described for all eclipses by the same curve*, provided the abscissæ are taken proportional to ρ, and the ordinates to ρ (hyp. A) or ρ^2 (hyp. B) ; the number ρ is simply proportional to the duration of the totality. These two fundamental curves F_1, F_2 are tabulated in Table I. ; moreover, we have

TABLE I.

The Functions F_1 and F_2,

and the intensity of the radiation for $u = 0{\cdot}40$, $m = 25$, $v = 1{\cdot}2$.

Total Eclipses.

t/ρ.	c/ρ.	F_1.	F_2.	J_1/I.	J_2/I.
2·0	3·0	1·5935	2·5276	0·521 ϱ/r	26·5 ϱ^2/r^2
1·0	2·0	0·6849	0·5435	0·224	5·69
0·5	1·5	0·2770	0·1102	0·0906	1·15
0·2	1·2	0·0776	0·0124	0·0254	0·130
0·1	1·1	0·0286	0·0023	0·0094	0·024
0·0	1·0	0	0	0	0

Annular Eclipses.

−2·0	−3·0	−4·7351	14·7511	1·55	154
−1·0	−2·0	−3·8265	8·8813	1·25	93·0
−0·5	−1·5	−3·4186	6·5557	1·12	68·6
−0·2	−1·2	−3·2192	5·3912	1·05	56·4
−0·1	−1·1	−3·1702	5·0400	1·04	52·8
0	−1·0	−3·1416	4·7124	1·03	49·3

given also the intensity of the radiation, J_1/I and J_2/I, for a mean value of the constants : $u = 0{\cdot}40$; $v = 1{\cdot}2$; $m = 25$. The use of the curves will be clear and may be described as follows :—

We consider first observations made upon the central line. For a particular eclipse, take the duration of the totality in seconds of time and the duration of the partial phase from first contact to centrality ; these are equivalent to 2ρ and $2r$. If the radiation is required at a moment t sec. before the beginning of the totality (or after its end), take the value of $\frac{t}{\rho}$ and look up in our table the corresponding ordinate J_1/I or J_2/I, according to whether the hypothesis A or B is assumed. If u, v, m are not assumed $0{\cdot}40$, $1{\cdot}2$, 25, the ordinates must simply be altered proportionally.

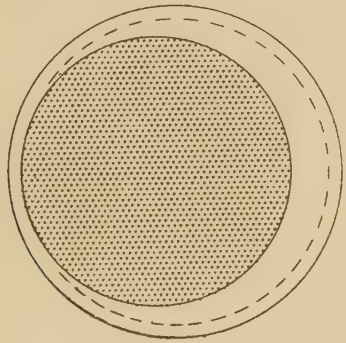


Fig. 2.

If the observations are not made upon the central line, c is no longer equal to $t + \rho$, but can be easily calculated at any moment.

3. *Annular Eclipses.*

This case is treated with the same formulæ as the total eclipses, only ρ is negative, and, moreover, in (2) and (4) the radicals must be taken with the negative sign.

During the annular phase, from 2nd to 3rd contact, we consider the excentric ring to be the sum of two parts, as explained by fig. 2. In the same way as above, we find the formulæ for the annular phase :

$$\text{Hypothesis A:}\quad \frac{J_1}{I} = -2uv\frac{\rho}{r}.$$

$$\text{Hypothesis B:}\quad \frac{J_2}{I} = mv\frac{\rho^2}{r^2}\left[1 + 2\frac{c}{\rho} + \frac{5}{2}\frac{c^2}{\rho^2}\right].$$

In the first case the total radiation is constant during the annular phase ; in the second case it varies in the proportion 3 to 2 (for $c = -\rho$ and for $c = 0$).

When an eclipse is nearly on the verge between total and annular, ρ being very small, it is advantageous to develop the formulæ (2) and (4) in terms of $\frac{\rho}{c} = \gamma$. They reduce to the following series:—

(5) Hypothesis A: $$\frac{J_1}{I} = \frac{2uv}{\pi}\frac{c}{r}\left(1 - \frac{\pi}{2}\gamma + \frac{\gamma^2}{2}\right).$$

(6) Hypothesis B: $$\frac{J_2}{I} = \frac{mv}{4}\frac{c^2}{r^2}\left(1 - \frac{8}{\pi}\gamma + 2\gamma^2 - \frac{1}{\pi}\gamma^3\right).$$

4. *Comparison with the Observations.*

In reviewing the numerous measurements of the radiation during eclipses, it appears that nearly all of them have been taken in such a way that they are of little use to science. Yet the very simplest measurements, even made by improvised apparatus, could have given useful results if a good method had been applied. We need measurements of the total solar radiation during the two or three minutes near totality, especially during the last ten or twenty seconds. A continuous record is still better. A high accuracy is not required; even errors of 50 per cent. are not very important. It is highly desirable to measure with the same instrument, before and after the eclipse, the total radiation of the sun. Selenium photometers must be avoided, because they have a considerable hysteresis; but all other photometric devices are practicable. If possible, the measurements should be made with a colour screen; but measurements in integrated white light or photographic or visual light are nevertheless also valuable.

For the sake of illustration, we will calculate the eclipse curves for two typical cases: an eclipse just on the verge annular-total, and an eclipse of five minutes' totality. In both cases we assume the interval between 1st contact and centrality to be 80 min. Table II. gives

TABLE II.

The Eclipse Curve for Two Typical Cases.

I. A very long total eclipse.

$2\varrho = 5$ min.; $2r = 80$ min.; $\frac{\varrho}{r} = \frac{1}{16}$.

t before 2nd Contact.	t/ρ.	J_1/I (Hypothesis A).	J_2/I (Hypothesis B).
120 sec.	0·80	1060	1340
90 ,,	0·60	725	690
60 ,,	0·40	420	266
30 ,,	0·20	159	51
20 ,,	0·13	88	15
10 ,,	0·07	34	3·5

TABLE II.—*continued.*

II. An only just total eclipse.

$\varrho = 0$; $2r = 80$ min.

t before Centrality.	c/r.		J_1/I (Hypothesis A).	J_2/I (Hypothesis B).
120 sec.		0·05	1525	1875
90 ,,	3/4.	0·05	1144	1055
60 ,,	1/2.	0·05	763	469
30 ,,	1/4.	0·05	381	117
20 ,,	1/6.	0·05	254	52
10 ,,	1/12.	0·05	127	13

the radiation compared with that of the uneclipsed sun. The main difference between hypotheses A and B is clear at once ; in the first case the radiation falls off according to a nearly straight line ; in the second case more or less as a parabola. The two radiations are nearly equal 95 sec. before totality ; then the curve B falls off much more rapidly, 10 sec. before totality it has only $\frac{1}{10}$ of curve A. Another result is very curious : *the determination of the distribution of the brightness by the eclipse method is just as accurate when made during an eclipse of long duration as during a very short one,* for the quotients between columns A and B are about the same in both cases.

The Eclipse of 1912.—This is the only case in which good observations are available. Julius and Moll * observed the eclipse at Maastricht (Holland) with a perfectly clear sky ; they measured the total radiation of all wave-lengths, one of them with a thermopile, the other with a bolometer. The second instrument was the quicker and its deflections were recorded by photography, so they are to be preferred for our purpose. The published curve is magnified with an epidiascope and very slightly smoothed out. The influence of the inertia of the apparatus may be eliminated nearly entirely ; for on the record we see also a point where a screen was put before the instrument, and we find that the deflection there falls off after the law $e^{-0\cdot74\tau}$, τ being the time in seconds. It is easily shown that when the radiation decreases during a long time at the rate of d units per second, the bolometer deflection is at every moment too high by $\frac{d}{0\cdot74}$, and that it is too low by the same amount when the radiation increases. After this correction was applied, the mean was taken of the intensities before and after the totality. At the moment of centrality, a radiation 4 remains, which is composed : 1° of the radiation of the extreme limb of the photosphere ; 2° of the radiation of the chromosphere and the corona ; we estimate this last part as 2. During the partial phase the moon covers about one-half of the chromosphere ; we therefore subtract everywhere 1, only one and two seconds before totality 2 is subtracted. The numbers thus corrected, given in Table III., column 2, are probably the best data known on the radiation near totality. The (uncorrected) values

* *Ap. J.*, **37**, 225, 1913.

obtained by the thermopile are given between brackets for comparison, and appear to confirm quite well those of the bolometer.

TABLE III.

The Integrated Radiation during the Solar Eclipse of 1912.

Time from Centrality.	Observed.		Calculated. A.	B.	$0{\cdot}98_5$A+0·40B.	C.	$0{\cdot}98_5$C+0·45B.
1 sec.	$8{\cdot}10^{-5}$		24	0·4	24	13	13
2 ,,	20		36	0·9	36	22	22
3 ,,	33		48	1·7	48	34	34
4 ,,	45		61	2·8	61	46	47
5 ,,	57		73	4·1	74	57	58
10 ,,	122	(102)	137	14	141	119	122
15 ,,	190		199	31	208		
20 ,,	274	(250)	264	54	282	244	261
30 ,,	434	(410)	390	120	432	371	416
40 ,,	597		517	211	592	495	578
50 ,,	765		644	326	764	623	755
60 ,,	940	(920)	765	470	941	747	940

This eclipse was annular but on the very verge of totality. From the data of Julius we find $2r = 4845$ sec., $2\rho = 0{\cdot}98$ sec. The radiation according to hypotheses A and B is calculated by the formulæ (5) and (6) and given in Table III., columns 3 and 4.

The comparison shows at once that hypothesis A is much more consistent with the observations than B. Still better is a combination of both: $0{\cdot}98_5\mathrm{A} + 0{\cdot}40\mathrm{B}$ giving a satisfactory agreement except for the last seconds (column 5).

Suppose the remaining deviation for the last seconds is not due to errors of observation, *e.g.* to the unsteadiness of the zero-point, but let it be real. For explaining this quicker decrease, we attempt to describe the brightness near the limb by

$$\text{Hypothesis C:} \quad i = \frac{p}{k^2} - \frac{p}{(k+z)^2},$$

z representing again the distance to the limb; p and k are constants, and $p/k^2 = u$; as soon as z becomes great, i tends towards the constant value u (fig. 3). In this case, too, the eclipse curve may be easily calculated. The radiation of a radial strip is

$$\int_0^z i\,dz = \frac{p}{k^2}z + \frac{p}{k+z} - \frac{p}{k}.$$

And the radiation of the whole crescent

$$J_3 = 2p\left(\int_0^{\phi^*} \frac{c\cos\phi - \rho}{k^2}\,d\phi + \int_0^{\phi^*} \frac{d\phi}{c\cos\phi - \rho + k} - \int_0^{\phi^*} \frac{d\phi}{k}\right)$$

$$- 2p\left(\frac{c}{k^2}\sin\phi^* - \frac{\rho}{k^2}\phi^* - \frac{\phi^*}{k} + \frac{2}{(-c-\rho+k)\sqrt{q}}\arctan\frac{t}{q}\right)$$

or $$2p\left(\frac{c}{k^2}\sin\phi^* - \frac{\rho}{k^2}\phi^* - \frac{\phi^*}{k} + \frac{1}{(c+\rho-k)\sqrt{-q}}\log\frac{\sqrt{-q}+t}{\sqrt{-q}-t}\right),$$

according to the case $c < k - \rho$ or to $c > k - \rho$; here $q = \dfrac{c-\rho+k}{-c-\rho+k}$,

and $\phi^* = \arccos\dfrac{\rho}{c}$ as before.

We attempt to put $k = 1$ sec., the other dimensions being also expressed in seconds of time corresponding with the movement of the moon during the eclipse (here 2422 sec. = solar radius). Consequently $p = 0{\cdot}4$, for u was assumed 0·4. The values of the radiation for this hypothesis C, which is only a slight modification of A, are given in Table III., column 6, expressed in parts of the total radiation of the

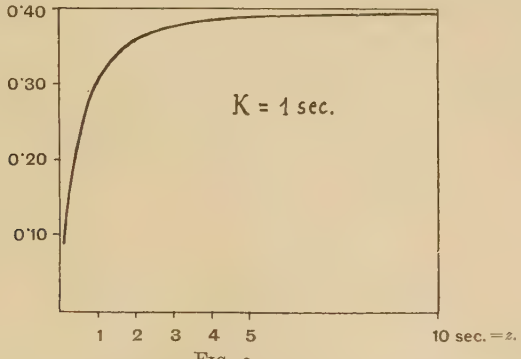


Fig. 3.

uneclipsed disc. It will be seen at once that the quicker decrease during the last few seconds is described much better. With $k = 1{\cdot}1$ an almost perfect agreement would probably have been reached.

On the whole, the eclipse curve during the last minute is best explained by a combination of the radiations B and C, according to the formula $0{\cdot}45\mathrm{B} + 0{\cdot}98_5\mathrm{C}$ (column 7), corresponding to: $u = 0{\cdot}39$; $m = 10$; $k = 1{\cdot}1$; or

$$i = 0{\cdot}39 + 10\frac{\Delta r}{r} - \frac{0{\cdot}39}{\left(1 + 2200\frac{\Delta r}{r}\right)^2},$$

in terms of the central intensity. The brightness in the marginal parts of the disc, represented by this formula, is drawn in fig. 4. A steep decrease towards zero, if real, is restricted to the last thousandth part of the radius.

During the same eclipse, Jouaust and De la Gorce at Clayes have made some simple visual measurements, at too long intervals from the totality, but which may be used all the same.*

* *C.R.*, **154**, 1141 (1912).

Time from Totality.	Observed.	A.	B.
1 min.	680	750	470
2 ,,	1410	1500	1880
3 ,,	1940	2250	4230

The evidence is conclusively in favour of A. For visual light we take $v = 1{\cdot}25$. The best agreement is reached for $u = 0{\cdot}35$, $m = 1$ (between one and two minutes).

The Eclipse of 1905.—Schwarzschild determined by photography the intensity of the last crescent at different moments shortly before totality, and found for 4550 A.U. : $u = 0{\cdot}17$, $m = 4{\cdot}8$. His beautiful

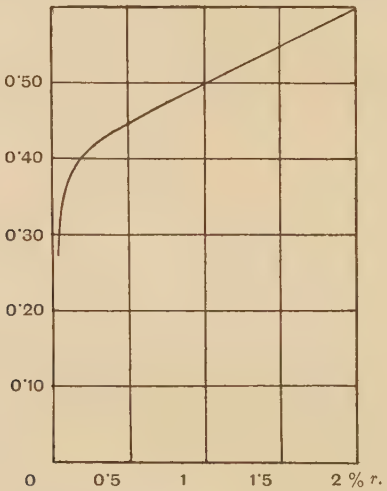


FIG. 4.

method resembles that of Julius and is also independent of the scintillation, but his measurements are only rough.*

The Eclipse of 1926.—Duration of the totality: 172 sec. The eclipse curve was determined for visual light by Stetson, Coblentz, Arnold, and Spurr.† The measurements were made somewhat too far from totality; the curve cannot be used for distances less than 43 sec., for there it is only an interpolation.

Time from Contact.	Observed.	A.	B.
43 sec.	149	301	92
86 ,,	440	740	680
172 ,,	1500	1480	4400

The best agreement between 43 sec. and 86 sec. is found for 0·44A + 0·16B, which means $u = 0{\cdot}17$ and $m = 4$, for $v = 1{\cdot}25$.

* *Mitt. Sternw. Gött.*, Teil 13, 1906.
† *Ap. J.*, **66**, 65, 1927.

General Conclusion.

Provisionally the following results are given by the eclipse method :—

Eclipse.	λ.	Marginal Brightness.	Slope.
1912	Integrated	0·40	10
1912	Visual	0·36	1
1926	,,	0·17	4
1905	4550 A.U.	0·17	5
(*Cf.* Vogel, *loc. cit.*	5700 A.U.	0·24	5)

The greater relative brightness at the limb for integrated light and the steeper slope are consistent with the former measurements for the more central parts of the disc. The results of Schwarzschild and of Stetson c.s. are in good agreement with each other and with the old results of Vogel. The results of Jouaust and De la Gorce are rough and could perhaps have been rejected, considering the amount of (false ?) light they recorded at the moment of totality.

From the existing data, we infer that the intensity at the sun's limb falls off suddenly. These data are not well suited for an exact determination of the marginal brightness, but better may readily be obtained ; only for integrated light are they satisfactory.

Utrecht,
Heliophysical Institute of the
Physical Laboratory :
1928 *November.*

A Photographic Chart of the Northern Milky Way. By C. Easton. (Plates 1, 2.)

Several years ago I combined the then available photographs of the Milky Way in a general " Photographic Chart " of the entire galactic zone. This photographic chart was used for the construction of the " circular " chart, given with a paper in the *Astrophysical Journal*, **37**, No. 2 ; but, of course, the particular projection and the very limited scale were rather serious drawbacks. Moreover, the general distribution of the galactic light was lost in printing. The four original drawings, each measuring about 17 by 10 inches, have never before been published. The accompanying plates are reduced reproductions of two of these sheets, representing the northern half of the Milky Way.

The stars have been inserted after the data furnished by A. Marth,* the distortion in Marth's projection being very slight. As it has been explained in my paper quoted above, the photographs furnished each a certain amount of detail, and the connecting of these details presented no great difficulties ; overlapping photographs, often by different authors, being available for almost the whole zone, at least for the central and most important parts of it, between + 10 and − 10 degrees of gal-

* " Galactic Longitude and Latitude of Stars in the Milky Way," *Monthly Notices*, **53**, 78, 384, 420.

actic latitude. In this manner the general picture of the Milky Way resulted with a minimum of bias on the part of the draughtsman. The contrast between the bright and dark patches and lanes has intentionally been somewhat exaggerated in drawing.

A detailed list of the principal photographs used in my drawings was given in *Ap. J.*, **37**, 106. A few photographs that could be of some use have been published since the original drawings were made, and before photographing the sheets a general revision was undertaken, but no important changes seemed indicated by the new material. In studying a particular region it will of course always be necessary to resort to the original photographs. The beautiful photographs taken by the late Professor Barnard * form the most recent addition to our collection. Unhappily it is again a *selection* only, and the northern part of the zone is but poorly represented in those splendid images of (rather restricted) portions of the great stellar agglomeration. *A systematic and homogeneous series of overlapping photographs, embracing the whole breadth of the entire zone*, is still badly wanted.

As it is, a " photographic chart " of the Milky Way can never give an exact representation of the relative brightness of its widely separated regions. I trust, however, that my charts may be of some use for the investigation of the general structure of the stellar system, as it depicts itself on the photographs.

Not much value should be given to the outlying streamers; but in the principal branches the wealth of detail in these two plates is far greater than in the " Photographic Milky Way," published by Dr. F. Goos, of Hamburg, in 1921, after a series of photographs taken by Professor Max Wolf.

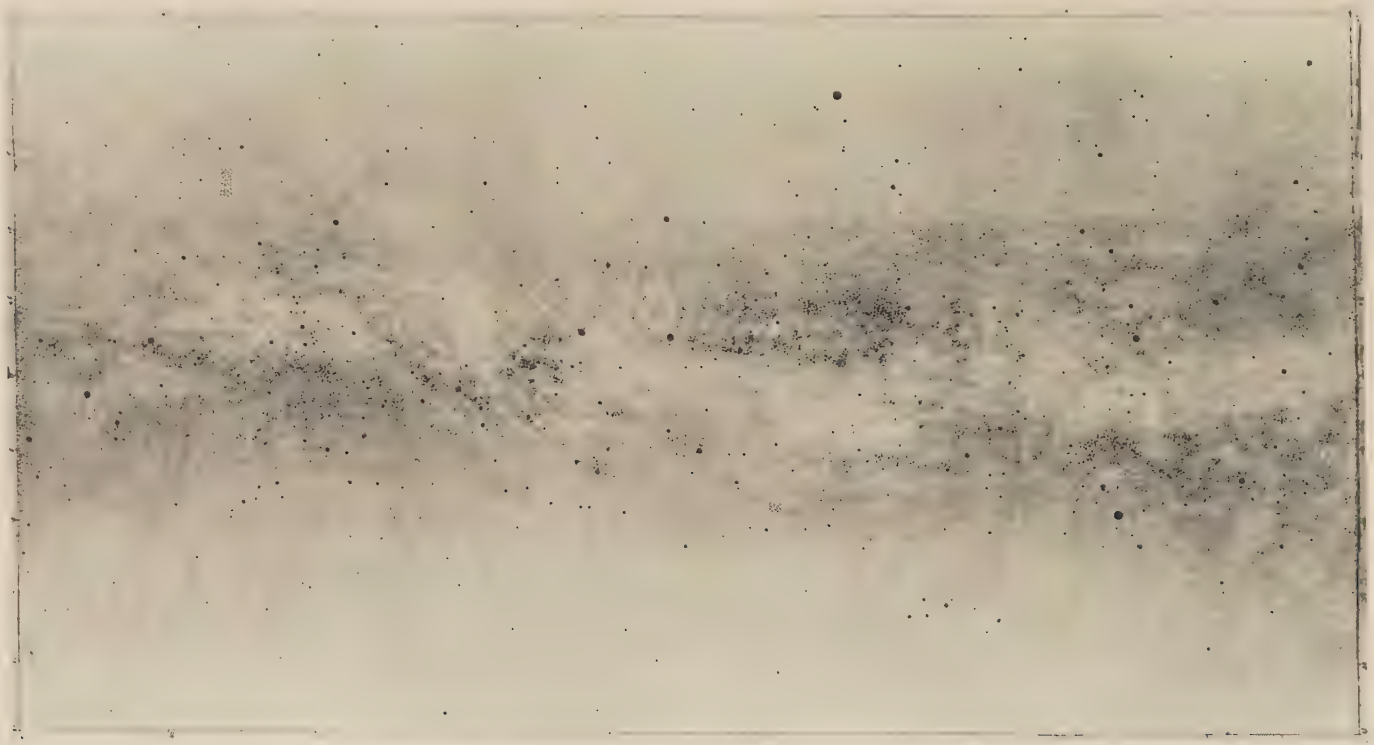
Some features of our image of the galaxy are very striking and undoubtedly real, *e.g.* the extensive system of streamers between δ *Cassiopeiæ*, the Pleiades, β *Tauri* and δ *Aurigæ*, converging to the central parts in *Auriga* and *Perseus* †—although, owing to the loss of detail, the regularity of those lateral offsets in *Perseus* and *Taurus* may have been slightly exaggerated on our chart. The great system of streamers in *Orion* seems to form a complement to it. It is perhaps significant that the lateral rifts separating those two systems and crossing the whole central region of the Milky Way in *Auriga* should lie at a distance of 180° from the complicated system of streamers in *Sagittarius* and *Scorpius*.

In a former publication (*loc cit.*, p. 109) I wrote : " It is obvious that the extraordinary chasms and bright and dark streams, discovered by Professor Barnard between 52 *Ophiuchi* and *Antares*, form but a part of a much more extensive system of interlacing branches—perhaps there are *two* systems present, seen in perspective and projected one

* " A Photographic Atlas of Selected Regions of the Milky Way," by E. E. Barnard, edited by E. B. Frost and M. J. Calvert, *Publ. Carnegie Inst. of Washington*, 1927.

† On the S. border of those streamers we find the extensive regions of absorbing matter, shown on the plates of Franklin Adams (partly also on Barnard's photographs just published) and studied by Sir F. W. Dyson, P. J. Melotte, A. Pannekoek, and others.

N

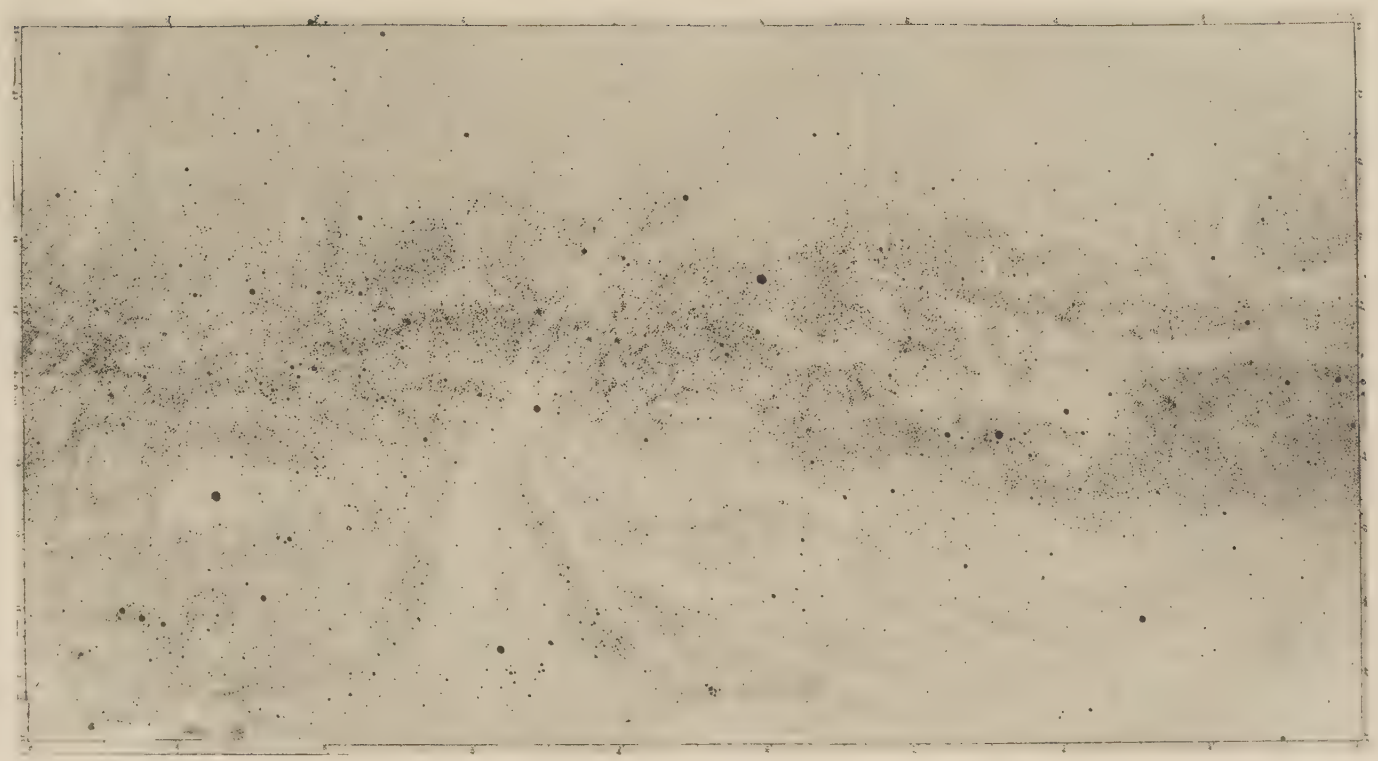


Julius Oppenheim, *photo.*

C. Easton, *del.*

0°—90° Gal. Long.

N



Julius Oppenheim, *photo.* C. Easton, *del.*

90° — 180° Gal. Long.

against the other—covering the whole region to the north of the galactic plane between η *Serpentis* and β *Centauri*, and certainly related to the principal (southern) branch of the Milky Way, though separated from it by rifts and cavities. It can hardly be by mere chance that most of the streams radiating from the gigantic assemblage of suns about σ and ν *Scorpii* are directed to the very brilliant patches of milky light in *Scutum* and *Sagittarius*. It is obvious that all these formations are interdependent and not merely the outcome of a chance projection; the unity in this apparent complexity is highly significant."

In making use of the accompanying photographs, the following should be borne in mind:

The photographic reduction of my original drawings, on so large a scale, due to Mr. Julius Oppenheim, the Hague, proved to be no easy matter, and some slight defects could not be avoided; they may be easily corrected by the reader. There is some shadowing in the outlying parts of Plate 1, also on Plate 2, near the Pleiades and to the north of Capella, and in the lower parts in *Monoceros*. The brightness of the *Aquila*-region (bright in the sky is of course dark on the plates) is somewhat overdone. The region between α *Cassiopeiæ* and γ *Andromedæ* appears too bright; on the contrary, the parts about γ, α, and ϵ *Cygni* (Plate 1) and around α *Orionis* and about γ *Geminorum* should show a greater intensity of galactic light.

The Revival of Activity in the Southern Hemisphere of Jupiter. By F. J. Hargreaves, B. M. Peek, M.A., and Rev. T. E. R. Phillips, M.A. (Plates 3–5.)

1. The purpose of this joint paper is to give an account of the recent revival of activity in Jupiter's southern hemisphere and the very remarkable phenomena which accompanied it in its earlier stages—that is, from its beginning shortly before the middle of August of the present year to the end of October. A great many observations have also been obtained, especially of late, by other observers, which will provide valuable material for a full discussion of developments which are still in progress. The observations herein described were secured by the authors working in concert at about the time when the disturbed region was rapidly extending itself round the planet and displaying features of a special character.

The following are particulars of the localities and instrumental equipment of the observers:—

T. E. R. Phillips.	Headley, near Epsom.	18-in. reflector and 8-in. refractor.
F. J. Hargreaves.	Kingswood, Surrey.	$6\frac{1}{2}$-in. reflector.
	Headley.	18-in. reflector.
B. M. Peek.	Herne, Kent.	$12\frac{1}{2}$-in. reflector.
	Solihull, Warwickshire.	„ „

The removal of B. M. Peek from Herne to Solihull, Warwickshire, involved an interruption in the work of this observer between September 19 and October 24.

Dates and Times.—In order to avoid the difficulty and inconvenience in plotting the observations which would arise from changing the date at midnight, the day is reckoned to begin—like the Julian day—at Greenwich mean noon.

2. *General Description of the Observations.*—(*a*) It may be recalled that during the apparition of 1925 certain well-known features of the planet's disc, viz. the Red Spot hollow, the southern component of the south equatorial belt, and the south tropical disturbance, disappeared as they had done in 1919. But, as was the case in the latter year, the Red Spot itself became more prominent, and in 1927 it regained something of its original redness. The conditions were unchanged when the planet was again observed in the early part of the present apparition. Since, however, the reappearance of the above-mentioned features had been marked in 1920 by certain abnormal phenomena (see *B.A.A. Memoirs*, vol. xxvi., part iv., pp. 77 *et seq.*, and *B.A.A.J.*, vol. xxxvii. p. 62), observers had been advised to be on the alert for any similar occurrences whenever the expected revival after the 1925 disappearance should set in.

(*b*) The beginning of the recovery seems to have been signalled by the appearance of a very dark spot in the latitude of the southern part of the south equatorial belt in longitude 128° ±(System II.). It was observed by Peek in the early morning following the night of August 12, and after a few days it was seen to be the centre of a disturbance which was extending itself both in the *following* and *preceding* directions. In the *f* direction a number of small black spots were successively formed which travelled in the latitude of the southern component of the belt at a speed altogether unprecedented in the records of Jovian observations. Five of these spots were at first distinguished, but a little later many others developed and showed the same rapidity of motion. On the whole, their drift was remarkably uniform, and since their longitudes were increasing at the rate of some 5° per day (the equivalent of about 3500 miles), they quickly approached the Red Spot which was then situated at about longitude 315° (System II.). What happened when conjunction occurred will be described later.

The table which follows gives a list of the spots best observed during the period covered by the paper, and the deduced rotation periods. Previous to this outbreak the longest rotation periods recorded of any Jovian feature (so far as known to the authors) were $9^h\ 57^m\ 58^s{\cdot}9$ and $9^h\ 57^m\ 52^s{\cdot}0$ shown by two dark south tropical spots during the revival of activity in 1920 above mentioned.

(*c*) In the *preceding* direction the disturbance took the form of a confused turbulence in a slightly lower latitude. The white and dark spots which were formed exhibited changes so rapid that in many cases it was impossible to identify the features when they were presented to view again after two rotations, but they appear to have been more or less stationary relatively to the longitudes of System II. Meanwhile

the *p* end itself, as indicated apparently by a rather small white spot at the *f* end of a rapidly moving break or gap in the northern component of the belt, travelled forward leaving the disturbed area in its wake.

TABLE.

No.	Longitude (System II.) on Sept. 20.	No. of Obs.	Rotation Period.	No.	Longitude (System II.) on Sept. 20.	No. of Obs.	Rotation Period.
	°		h m s		°		h m s
1	163	6	9 58 40·5	6	238	17	9 59 17·1
2	173	7	9 58 50·8	7	265	17	9 59 28·6
3	184	10	9 58 51·1	8	285	22	9 59 19·3
4	199	10	9 59 13·5				
5	210	19	9 58 47·8		Mean rotation period=9 59 3·6		

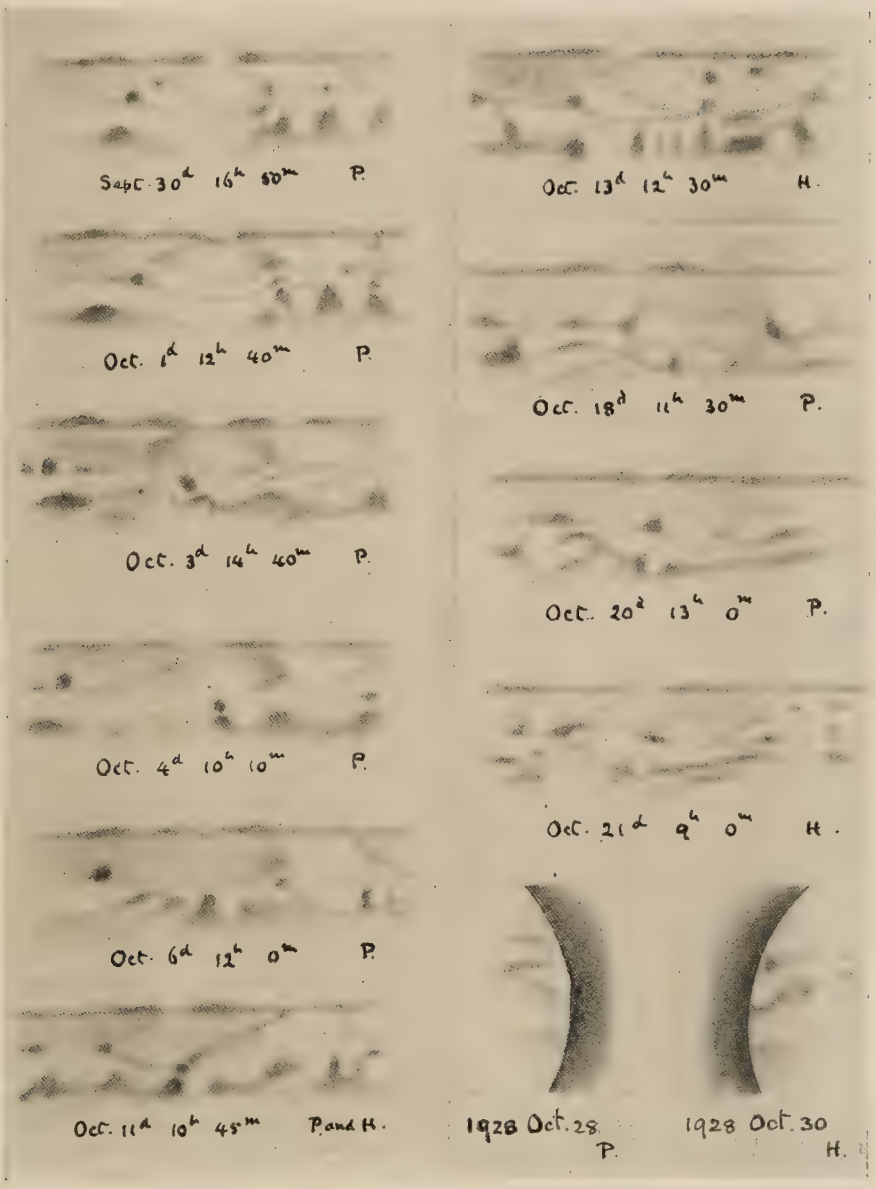
(*d*) The relative motion of the two ends of the disturbance was such that by September 22 they overlapped in longitude, and the disturbed area thus completely encircled the planet. Owing, however, to the difference in latitude there was no actual collision, though eventually the broadening of the disturbed regions led to the formation of a wide belt in which an amazing amount of intricate detail was shown. Moreover, the *preceding* half of the disturbance, which seems to have been forced northwards about the longitude of the Red Spot, extended itself over the southern part of the equatorial zone, producing in that region a very remarkable and unfamiliar appearance with large brilliantly white spaces and irregular dark areas and spots. And, without going here into the details of later developments, it may be stated that the observations have indicated an intimate physical connection between the turbulent features of the zone and the dark crests bordering the north equatorial belt on its south side. The nature of this connection will doubtless appear when all the available observations are reduced and discussed. A new belt now appears to be forming very close to the equator.

(*e*) Something must now be said as to the reactions between certain of the objects above described and the Red Spot. The small dark spots exhibiting the abnormally rapid motion in the *following* direction referred to in section 2*b* were situated in the same latitude as the northern part of the Red Spot, and a careful look-out was maintained at the critical times to determine what happened as they successively came into conjunction with it. Clouds prevented observations of the leading spot, but there is reason to believe that it faded out before conjunction. Several of the others were satisfactorily followed, and it was seen that the more advanced members of the series, which had become progressively larger, were definitely deflected northwards by the Red Spot, but that in their passage through the narrow channel between its northern border and the southern component of the south equatorial belt they were flattened into short streaks. They

survived their passage, however, although very shortly afterwards they either faded out altogether or were lost in the turbulent mass of material developing in the region on the other side of the Red Spot. The later members of the series, some of which became greatly elongated in longitude, actually traversed the northern edge of the Red Spot. These, too, died out or were lost to view soon after their passage. Meanwhile the extension and development of the turbulent region had resulted in the re-formation of the Red Spot hollow, but this was accompanied by a marked deterioration in the definiteness and intensity of the Red Spot itself. The latter also underwent some strange changes of form, and on October 11 a dark streak was seen extending in the $np - sf$ direction across the following part of the spot. On October 18 a dusky streak was observed stretching right across the hollow. Some very beautiful photographs taken in blue light one rotation later at the Lowell Observatory—showing the Red Spot much darker than it appeared visually—disclosed that this streak was the northern edge of the Red Spot, which seems to have been temporarily displaced southwards, leaving a wide channel between it and the southern component of the south equatorial belt. Plate 3 contains illustrations of the extensive and sudden changes observed in this region.

Colour Changes.—Some interesting colour changes were observed in the region dealt with in this paper. Mention has already been made (2*a*) of the revived redness of the Red Spot, but as seen in the 18-in. reflector there was a very definite pinkish glow about the object which differentiated its redness from that of its surroundings. But the colour faded considerably—though with fluctuations—as the hollow reappeared and the spot itself became much less conspicuous.

For at any rate some considerable portion of the time during which the usual south tropical markings had been absent the region included between the south temperate belt and the north component of the south equatorial belt had been brilliantly white. This appearance is intensified in ultra-violet photographs such as those taken by Dr. W. H. Wright at the Lick Observatory. In 1927, however, attention had been drawn—especially by M. Antoniadi at Meudon—to the redness of the southern portion of the south tropical zone p and f the Red Spot. The same phenomenon was very marked at the beginning of the present apparition, but later a reddish or tawny hue spread over the entire zone and became specially intense in the longitudes p the Red Spot. At the same time the northern component of the south equatorial belt was, in certain longitudes, strongly blue, and on September 19 the following note was made : " The blue of the S.E.B_N. as far as the f end of the dark portion [that is, the p end of the gap referred to in 2*c*] is wonderful, and the colour seems to overflow into the white region on its south side. The contrast between this and the reddish colour of the south tropical zone and the strong ochre tone of the equatorial zone is very striking " (T. E. R. P.). At the end of the month the blue of this dark part of the belt had practically disappeared, the impression being that it was obliterated by an encroachment of the strong tawny line of the equatorial zone which appeared


Sept. 30d 16h 50m P.
Oct. 1d 12h 40m P.
Oct. 3d 14h 40m P.
Oct. 4d 10h 10m P.
Oct. 6d 12h 0m P.
Oct. 11d 10h 45m P and H.
Oct. 13d 12h 30m H.
Oct. 18d 11h 30m P.
Oct. 20d 13h 0m P.
Oct. 21d 9h 0m H.
1928 Oct. 28 P.
1928 Oct. 30 H.

to be drifting southwards. Other portions of the belt remained blue for a time, but it was remarked after the observation of a complete rotation of the planet on October 12 that "the blue colour of the northern component of the south equatorial belt seems to have almost, if not entirely, disappeared" (T. E. R. P.). On October 13 this component was noted as "narrow, brown in colour, and shades off imperceptibly into the equatorial zone" (F. J. H.). The region between the two components of the south equatorial belt continued for some time to be a very bright bluish white.

During the first half of October the northern component of the belt, in the regions still undisturbed, was remarkable for its visibility near the limbs of the planet; it was very much less obscured by limb-darkening than the other belts.

Irradiating Spot.—A phenomenon of quite unusual interest was observed at the end of the period covered by this paper, viz. a white spot in latitude slightly south of the northern component of the south equatorial belt, which appeared as a bright projection or irradiating spot at the limbs but was much less conspicuous, or unidentifiable, when well on the disc (Plate 3). It was seen first at Headley at the *p* limb on October 24 from $15^h\ 40^m$ to $16^h \pm$. The next evening it was detected quite independently at Kingswood at the *f* limb and followed from $7^h\ 28^m$ to $7^h\ 50^m$, and again at Headley at the *p* limb after traversing the disc from $11^h\ 18^m$ to $11^h\ 45^m$. Further observations were obtained at Headley and Kingswood on October 28, and at Solihull, as well as at Headley and Kingswood on October 30. The object then appeared to be extending itself over a larger area, and it soon lost its distinctive character. The observations obtained indicate that it was visible when more than 70° from the central meridian, and that it was rotating approximately at the rate of System I. The object was similar to those observed in 1919 and described in *B.A.A. Memoirs*, vol. xxvi., part iii., pp. 59 *et seq.*, but slightly further south.

3. *Presentation of the Observations.*—In the presentation and discussion of such very remarkable and in some respects unprecedented phenomena as the planet has lately exhibited it seems important that the observations should be given in considerable detail so as to make it possible for the identifications and conclusions to be checked. Probably the most convenient and useful way of doing this is to give the observations in chart form, and Plates 4 and 5 are appended for this purpose. An explanation of these plates is given in the following paragraphs:—

(*a*) Plate 4 is a pictorial representation of the progress of the disturbance, compiled from sketches made at the telescope in conjunction with determinations of the longitudes (according to System II.) of the markings.

The strip opposite each date comprises a latitude-zone extending from the northern edge of the northern component of the south equatorial belt to the northern edge of the south temperate belt. The longitude range is greater than 360°, in order to permit the Red Spot region to be repeated.

Round brackets ( ) indicate regions which were not observed at all, and square brackets [ ] indicate that no longitude determinations were made of the features shown between the brackets.

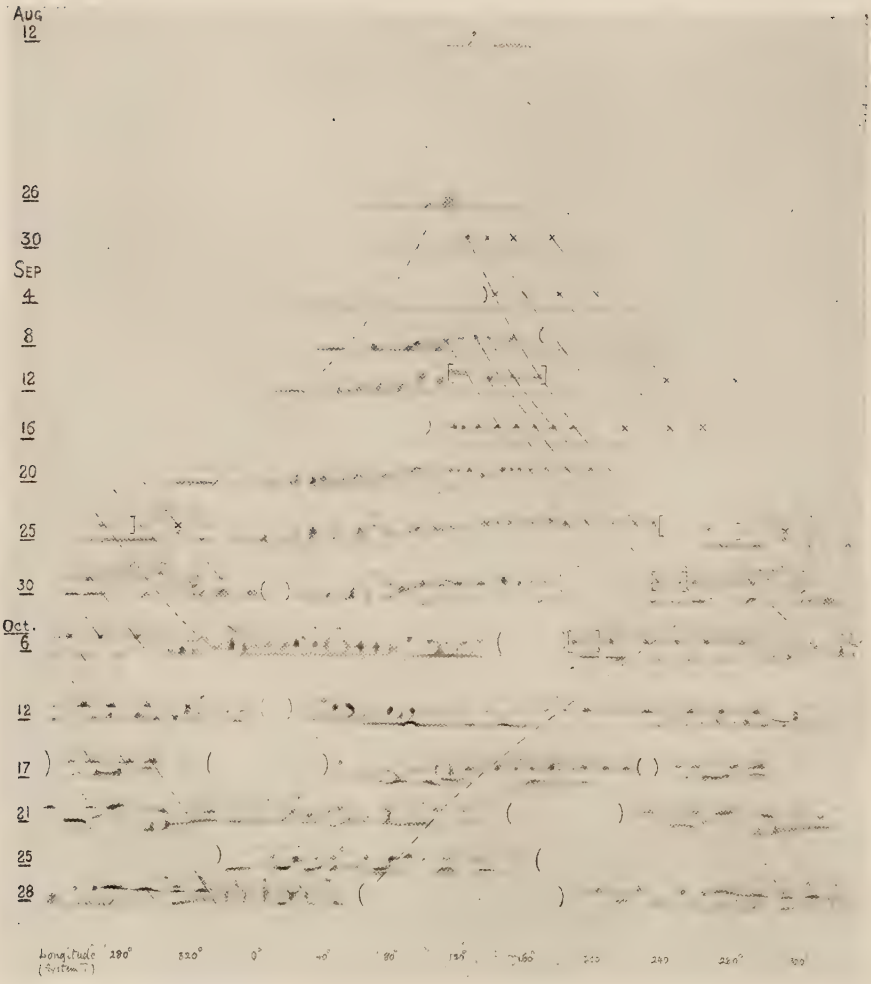
The symbol X indicates dark objects, the longitudes of which were determined but of which no sketches were made. In one or two cases the symbol X is used in a different sense. Since the ordinates of the diagram have to serve for latitude as well as time, a change of latitude, such as occurred when the rapidly moving dark spots passed the Red Spot, gives the appearance of a change of motion in longitude. In such cases the position which the spot would have occupied if its latitude had not changed is marked by an X.

Less than half of all the observations made are included in the diagram, because the inclusion of all of them would have resulted in a very long figure unsuitable for reproduction. The choice of dates has been made in such a way as to present as far as possible a regular, complete, and representative series of observations up to the time when the preceding end of the disturbance had completed a circuit of the planet with respect to System II., and the two halves of the disturbance had begun to lose their distinctiveness.

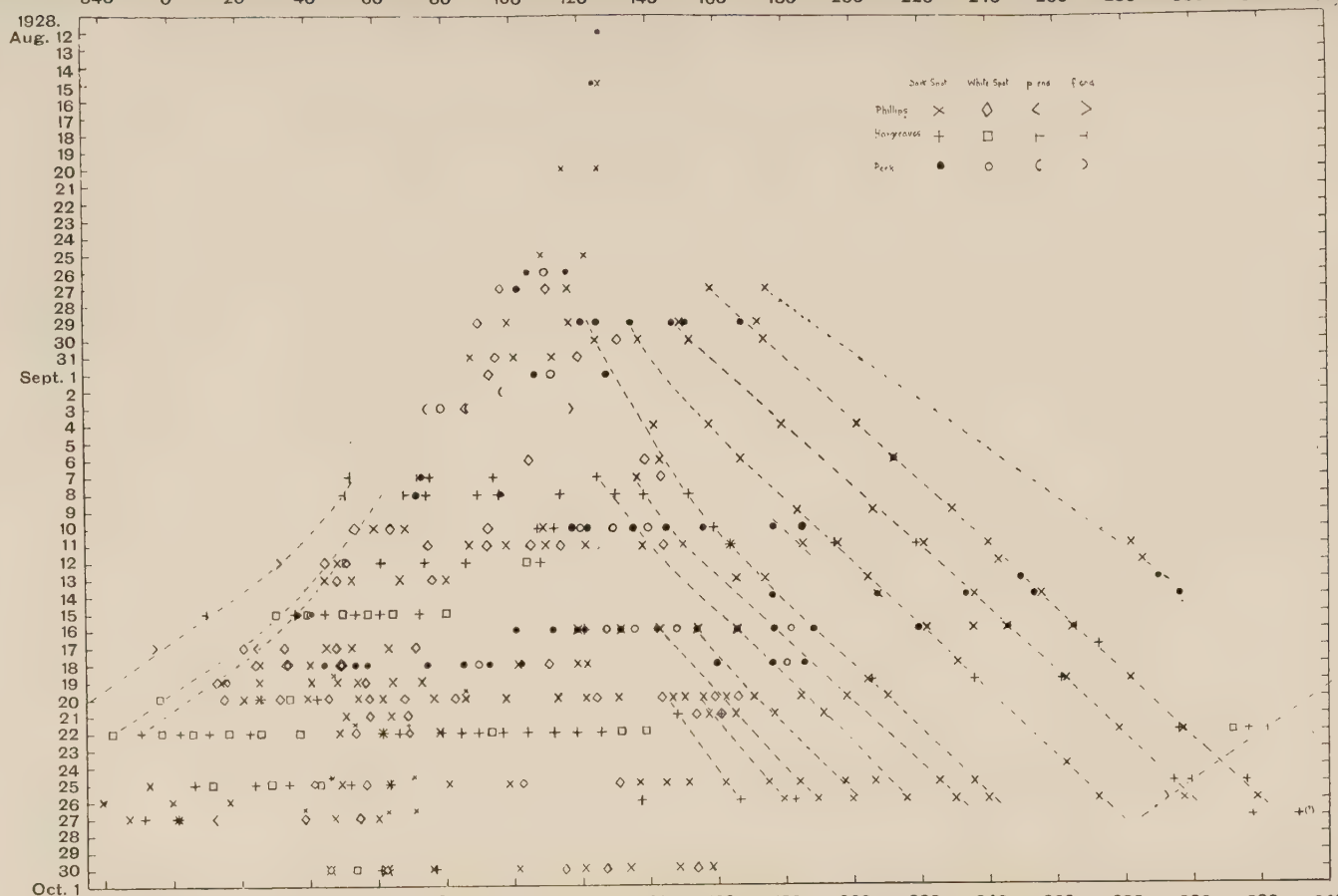
The dual character of the disturbance (2 *b*, *c*) will be apparent from this diagram—on the one hand the emission from the original centre of disturbance of a succession of dark spots moving rapidly in the following direction, and on the other hand the formation of a region of confused turbulence in the wake of the rapidly moving gap in the northern component of the south equatorial belt. A change will be noted in the character of the preceding part of the disturbance after it had spread beyond the Red Spot. The light and dark areas of which it consisted then became less strongly marked and less well-defined, and also occupied a rather more northerly zone of latitude, encroaching on the equatorial zone.

(*b*) Plate 5 contains a chart of all the longitude determinations in the region of the south equatorial belt and south tropical zone (apart from the Red Spot) obtained by the authors of the paper between the beginning of the revival and the end of September, *i.e.* about a week after the two ends of the disturbance had overlapped. The symbols used are explained on the chart. The longitudes are those of System II., of which the rotation period is $9^h\ 55^m\ 40^s{\cdot}632$. Lines of identification are drawn through observations relating to the same objects, and it will be noted that, although there is no room for doubt as regards the black spots at the *f* end, or the gap and white spot at the *p* end of the disturbance, it is impossible to trace the individual features elsewhere with any certainty for more than a few days together. But a comparison of this chart with that of Plate 4 appears to indicate that there was no systematic drift relatively to the adopted system of longitudes.

It may be mentioned that on several occasions changes were strongly suspected in certain objects in this region while under observation, and on October 13 a pronounced change of form was definitely noted after an interval of only forty minutes.



1928 Region of S. Equatorial Belt and S. Tropical Zone.


340 0 20 40 60 80 100 120 140 160 180 200 220 240 260 280 300 320 340
1928.
Aug. 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31
Sept. 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
Oct. 1
Dark Spot
White Spot
Phillips
Hargreaves
Perk

Determination of the Constants of the Position of an Orbit from its Ecliptic Elements. By Prof. Th. Banachiewicz.

(*Communicated by L. J. Comrie.*)

Denote, as ordinarily, by P_x, P_y, P_z, Q_x, Q_y, Q_z, the six direction cosines of the major and minor axes of a heliocentric orbit. These six cosines, introduced into theoretical astronomy by Gibbs,* together with the three direction cosines R_x, R_y, R_z of a normal to the plane of the orbit (which, apart from the check they give of $P_x \ldots Q_z$, are useful in computing the perturbations, as well as the differential corrections of an orbit †), are expressed here by formulæ of a new kind, involving the ecliptic elements ω, i, Ω, and the obliquity ϵ of the ecliptic.

In what follows there are used certain mathematical forms, the "matrices-cracovians," designated by the symbol $\{\}$ and differing from the Cayley matrices in the definition of the products. The product of two matrices-cracovians is defined as a matrix-cracovian whose elements in the ith column and kth row is the sum of the products of corresponding (*i.e.* belonging to the same rows) members of the ith *column* of the first factor and the kth column of the second factor. If the members of the ith column of $\{a\}$ are $a_1, a_2, \ldots a_m$, and the members of the kth column of $\{b\}$ are $b_1, b_2, \ldots b_m$, then the element in the ith column and kth row of the product $\{a\} \,.\, \{b\}$ is

$$a_1b_1 + a_2b_2 + \ldots + a_mb_m.$$

The product $\{a\} \,.\, \{b\} \,.\, \{c\}$ of three matrices-cracovians is to be taken as $[\{a\} \,.\, \{b\}] \,.\, \{c\}$; the product $\{a\} \,.\, \{b\} \,.\, \{c\} \,.\, \{d\}$ as $[\{a\} \,.\, \{b\} \,.\, \{c\}] \,.\, \{d\}$, and so on for more factors. A change of the order of factors in multiplying is not permissible, as it changes the elements of the product.

1. We have ‡

$$\begin{Bmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{Bmatrix} = \mathbf{r}(\omega) \,.\, \mathbf{p}(-i) \,.\, \mathbf{r}(-\Omega) \,.\, \mathbf{p}(-\epsilon) \qquad (1)$$

where $\mathbf{r}$ and $\mathbf{p}$ are the rotational matrices-cracovians (see *Circ. Obs. Crac.*, **17**):

$$\mathbf{r}(n) = \begin{Bmatrix} \cos n & -\sin n & 0 \\ \sin n & \cos n & 0 \\ 0 & 0 & 1 \end{Bmatrix} \qquad \mathbf{p}(n) = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \cos n & -\sin n \\ 0 & \sin n & \cos n \end{Bmatrix} \qquad (2)$$

Check:

$$\begin{Bmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{Bmatrix}^2 = \begin{Bmatrix} 1 & . & . \\ 0 & 1 & . \\ 0 & 0 & 1 \end{Bmatrix} \qquad (3)$$

* Gibbs, *On the Determination of Elliptical Orbits*, Washington, 1889.
† *Circ. Obs. Crac.*, **17**, 5. ‡ *Ibid.*, **17**, 6.

This check is, however, insensible to an error in the argument with which the sines and cosines are taken from the tables.

The formula (1), as restricted to the *six* constants $P_x \ldots Q_z$, is only another form of the known expressions,* differing from them, however, in that

(*a*) It is obtained instantly from a mere inspection of the polygonal line on the sphere;

(*b*) It is much shorter and it contains in itself a ready and rational scheme for the numerical calculation.

2. An alternative method of calculating $P_x \ldots R_z$ is the following:—

$$c = \sqrt{2}\cos\tfrac{1}{2}\epsilon \qquad t = \tan\tfrac{1}{2}\epsilon$$

$$\begin{Vmatrix}\kappa\\ \lambda\\ \mu\\ \nu\end{Vmatrix} = \begin{Vmatrix} c\cos\tfrac{1}{2}i\cos\tfrac{1}{2}(\Omega+\omega)\\ c\sin\tfrac{1}{2}i\cos\tfrac{1}{2}(\Omega-\omega)\\ c\sin\tfrac{1}{2}i\sin\tfrac{1}{2}(\Omega-\omega)\\ c\cos\tfrac{1}{2}i\sin\tfrac{1}{2}(\Omega+\omega)\end{Vmatrix} \begin{Vmatrix} 1 & t & 0 & 0\\ -t & 1 & 0 & 0\\ 0 & 0 & 1 & t\\ 0 & 0 & -t & 1\end{Vmatrix} \qquad (4)$$

$$a = 1 - (\lambda^2+\mu^2+\nu^2) = \kappa^2 - 1 \text{ (check).}$$

Then

$$\begin{Vmatrix}P_x\\ Q_y\\ R_z\end{Vmatrix} = \begin{Vmatrix}a+\lambda^2\\ a+\mu^2\\ a+\nu^2\end{Vmatrix}, \qquad \begin{Vmatrix}Q_z & R_x & P_y\\ R_y & P_z & Q_x\end{Vmatrix} = \begin{Vmatrix}\mu\nu & \nu\lambda & \lambda\mu\\ \kappa\lambda & \kappa\mu & \kappa\nu\end{Vmatrix} \begin{Vmatrix}1 & 1\\ 1 & -1\end{Vmatrix} \qquad (5)$$

The formulæ (4) and (5) are derived from a general transformation of an expression of the form (1), given in the *Sprawozdania* of the Polish Academy of Sciences, December 1927, and to be expounded fully in the *Acta Astronomica*. This transformation applies equally to other problems of spherical and theoretical astronomy. It would, of course, be easy to verify that the expressions (1) and (5), written in the ordinary way, are identical.

3. How many multiplications are to be performed in calculating the orbital constants in the above methods?

In using equation (1) the number of multiplications is $4+8+8 = 20$ for the six constants $P_x \ldots Q_z$, and $4+10+12 = 26$ for the nine constants $P_x \ldots R_z$. In using (4) and (5) this number is $2+4+4+9 = 19$ in both cases.

It appears, therefore, that the second method is somewhat shorter; its interest lies also in the fact that it gives an independent check of the results obtained by the first method.

The *Rocznik Astronomiczny Obserwatorjum Krakowskiego*, Supplem. Intern., No. 7, for 1929, will contain for different epochs the values of $\sqrt{2}\cos\frac{1}{2}\epsilon$ and $\tan\frac{1}{2}\epsilon$, calculated on the basis of Newcomb's precessional elements according to the numerical theory of the writer.†

* *Journal of the British Astronomical Association*, **32**, 232.

† Expressions analytiques pour le calcul de la précession, Cracovie, 1924.

For 1925·0 $c = 1{\cdot}3847\ 0732$ $t = 0{\cdot}2075\ 3640$
„ 1950·0 $c = 1{\cdot}3847\ 1548$ $t = 0{\cdot}2075\ 0679$
and
for 1929·0 $c = 1{\cdot}3847\ 0862$ $t = 0{\cdot}2075\ 3166$

Example.—Comet 1925 *c* (Orkisz). (2nd method)

$$\left.\begin{aligned} \omega &= 35^\circ{\cdot}8669 \\ \Omega &= 318{\cdot}0119 \\ i &= 99{\cdot}9417 \end{aligned}\right\} 1925{\cdot}0$$

(G. Merton, "Method for the Determination of Orbits," *M.N.*, **85**, 717.)

$\frac{1}{2}(\Omega+\omega)$	$176^\circ{\cdot}9394$	cos − 0·99857	sin + 0·05339
$\frac{1}{2}(\Omega-\omega)$	141·0725	cos − 0·77794	sin + 0·62834
$\frac{1}{2}i$	49·9708	cos + 0·64318	sin + 0·76572

$c = 1{\cdot}3847\ 0732$ $\quad c \cos \frac{1}{2}i\ 0{\cdot}89062$
$t = 0{\cdot}2075\ 3640$ $\quad c \sin \frac{1}{2}i\ 1{\cdot}06030$

Quantities printed without decimals are in units of the 5th decimal.

$$\begin{Bmatrix} \kappa \\ \lambda \\ \mu \\ \nu \end{Bmatrix} = \begin{Bmatrix} -88935 \\ -82485 \\ +66623 \\ +04755 \end{Bmatrix} \begin{Bmatrix} 1 & 0{\cdot}207536 & 0 & 0 \\ -0{\cdot}207536 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0{\cdot}207536 \\ 0 & 0 & -0{\cdot}207536 & 1 \end{Bmatrix} = \begin{Bmatrix} -71816 \\ -100942 \\ +65636 \\ +18582 \end{Bmatrix}$$

κ^2 51575
λ^2 101893
μ^2 43081
ν^2 3453

$\mu\nu$	$\nu\lambda$	$\lambda\mu$	Q_z	R_x	P_y
$\kappa\lambda$	$\kappa\mu$	$\kappa\nu$	R_y	P_z	Q_x

$$\begin{Bmatrix} +12196 & -18757 & -66254 \\ +72493 & -47137 & -13345 \end{Bmatrix} \begin{Bmatrix} +1 & +1 \\ +1 & -1 \end{Bmatrix} = \begin{Bmatrix} +84689 & -65894 & -79599 \\ -60297 & +28380 & -52909 \end{Bmatrix}$$

$a = -0{\cdot}48427 = (-0{\cdot}48425)$

Taking for a −0·48425 :

$P_x = +0{\cdot}53468$	$Q_y = -0{\cdot}05344$	$R_z = -0{\cdot}44972$	
+0·53468	−0·05344	—	(Merton)
$Q_z = +0{\cdot}84689$	$R_x = -0{\cdot}65894$	$P_y = -0{\cdot}79599$	
+0 84687	—	−0·79597	„
$R_y = -0{\cdot}60297$	$P_z = +0{\cdot}28380$	$Q_x = -0{\cdot}52909$	
—	+0·28380	−0·52910	„

Observatory, Cracow:
1928 *October* 23.

Occultations of Stars and Planet by the Moon, observed at the Nizamiah Observatory, Hyderabad, during the period 1927 December–1928 May.

(Communicated by the Director.)

No.	Date		Mag.	Phase.	Greenwich Mean Time. h m s	Observer.
	1927.					
1	Dec. 3	A.G. Strassburg 34	8·2	D	13 17 23·4	B
2	8	B.D. 22°·811	9·5	D	17 8 24·7	M
3	8	B.D. 22°·812	8·8	D	17 11 33·7	M
4	8	B.D. 22°·813	9·7	D	17 30 26·2	B
5	13	η Leonis	3·6	D	19 52 1·4	M
6	13	η Leonis	3·6	R	20 54 25·1	M
7	20	Venus	..	R	5 23 5·7	M
	1928.					
8	Jan. 5	121 Tauri	5·1	D	15 32 47·0	M
9	6	ε Geminorum	3·2	D	18 1 26·6	M
10	6	ε Geminorum	3·2	R	19 26 15·1	M
11	7	$\varkappa$ Geminorum	3·6	D	16 37 4·4	M
12	7	$\varkappa$ Geminorum	3·6	R	18 3 19·9	M
13	Feb. 6	η Leonis	3·6	R	15 23 2·5	M
14	14	ω Ophiuchi	4·5	D	23 15 51·6	M
15	15	ω Ophiuchi	4·5	R	0 11 24·1	M
16	26	A.G. Leipzig I. 891	9·0	D	13 34 58·7	B
17	26	A.G. Leipzig I. 892	9·5	D	13 35 5·7	B
18	Mar. 25	A.G. Berlin 979	9·4	D	13 54 50·5	B
19	25	A.G. Berlin 983	9·1	D	14 4 1·5	B
20	25	A.G. Berlin 981	8·2	D	14 7 56·3	B
21	25	A.G. Berlin 986	9·2	D	14 34 47·4	B
22	26	B.D. 21°·679	9·5	D	13 46 10·8	B
23	26	B.D. 21°·681	9·5	D	14 14 17·6	B
24	Apr. 1	46 Leonis	5·8	D	18 58 9·0	M
25	1	46 Leonis	5·8	R	20 6 19·1	M
26	28	42 Leonis	6·1	D	20 3 41·2	M
27	May 7	θ Ophiuchi	3·3	D	21 58 56·5	M
28	7	θ Ophiuchi	3·3	R	22 52 4·6	M
29	22	A.G. Cambridge E 3689	7·2	D	14 23 54·4	M
30	22	A.G. Cambridge E 3685	8·3	D	14 38 10·4	M
31	22	A.G. Cambridge E 3692	7·2	D	14 53 17·4	M
32	22	A.G. Cambridge E 3697	8·2	D	14 58 41·9	M
33	22	A.G. Cambridge E 3701	8·3	D	15 8 29·9	M
34	22	A.G. Cambridge E 3715	8·5	D	15 15 27·4	M

Instrument used : 15-inch Grubb refractor throughout, except for No. 26, which was observed with a 3-inch portable telescope.

Observers :

B = T. P. Bhaskaran.
M = M. K. Bappu.

Notes.

2, 3, 4 : observed during eclipse of the moon ; sky hazy.
4, 7, 25 : probably slightly late.
5 : gliding ; disappearance at the bright limb.
6 : reappearance at the dark limb.
29 : time slightly uncertain.
7, 21, 27, 28 : seen through clouds.

Nizamiah Observatory,
Hyderabad (Deccan),
India.